Multiprocessor Scheduling II: Global Scheduling

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Global Scheduling

- We will only focus on identical multiprocessors in this module.
- The system has a global queue.
- A job can be migrated to any processor.
- Priority-based global scheduling:
  - Among the jobs in the global queue, the M highest priority jobs are chosen to be executed on M processors.
  - Task migration here is assumed no overhead.
  - Global-EDF: When a job finishes or arrives to the global queue, the M jobs in the queue with the shortest absolute deadlines are chosen to be executed on M processors.
  - Global-FP, Global-DM, Global-RM: When a job finishes or arrives to the global queue, the M jobs in the queue with the highest priorities (defined by fixed-priority ordering, deadline-monotonic strategy, or rate-monotonic strategy) are chosen to be executed on M processors.
- Pfair scheduling, and the variances (not discussed in this lecture).

Good News for Global Scheduling

- McNaughton's wrap-around rule for $P|pmtn|C_{\max}$ on M processors (historically, task migration is also called task preemption in the literature)
  - Compute $C_{\max}$ as $\max\{\max_{\tau_i \in \tau} C_i, \frac{\sum_{\tau_i \in \tau} C_i}{M}\}$
    - Assign the tasks according to any order from time 0 to $C_{\max}$
    - If a task's processing exceeds $C_{\max}$, the task is migrated to a new processor from time 0
    - Repeat the assignment of tasks until all the tasks are assigned
  - The resulting schedule minimizes $C_{\max}$

R. McNaughton. Scheduling with deadlines and loss functions. Management Science, 6:1-12, 1959.

Weakness of Partitioned Scheduling

- Restricting a task on a processor reduces the schedulability
- Restricting a task on a processor makes the problem $\mathcal{NP}$-hard
- The $\mathcal{NP}$-completeness for EDF does no hold any more if the migration has *no overhead*.
  - Proportionate Fair (pfair) algorithm introduced by Baruah et al. provides an optimal utilization bound for schedulibility
  - A task set with implicit deadlines is schedulable on M identical processors if the total utilization of the task set is no more than M.
  - The idea is to divide the time line into quanta, and execute tasks proportionally in each quanta.
  - It has very high overhead.
  - There are several variances to reduce the overhead.

Sanjoy K. Baruah, N. K. Cohen, C. Greg Plaxton, Donald A. Varvel: Proportionate Progress: A Notion of Fairness in Resource Allocation. Algorithmica 15(6): 600-625 (1996)

Bad News for Global Scheduling

For Global-EDF or Global-RM, the least upper bound for schedulability analysis is at most 1.

Input:

$M + 1$ tasks:

- One heavy task τ_k: $D_k = T_k = C_k$
- M light tasks τ_is: $C_i = \epsilon$ and $D_i = T_i = C_k - \epsilon$, in which ϵ is a positive number, very close to 0.

Sudarshan K. Dhall, C. L. Liu, On a Real-Time Scheduling Problem, OPERATIONS RESEARCH Vol. 26, No. 1, January-February 1978, pp. 127-140.

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Result:

The M light tasks (with higher priority than the heavy task) will be scheduled on M processors. The heavy task misses the deadline even when the utilization is $1 + M\epsilon$.

Sudarshan K. Dhall, C. L. Liu, On a Real-Time Scheduling Problem, OPERATIONS RESEARCH Vol. 26, No. 1, January-February 1978, pp. 127-140.

Gold Approach: Resource Augmentation

- The bad news on the least upper bound was very important in 80's, since the research in this direction suffered from the so called "Dhall effect".
- With resource augmentation, by Phillips et al., the "Dhall effect" disappears
  - For Global-EDF, the resource augmentation factor by "speeding up" is $2 - \frac{1}{M}$.
  - That is, if a feasible schedule exists on M processors, applying Global-EDF is also feasible on M processors by speeding up the execution speed with $2 - \frac{1}{M}$.
  - We will focus on schedulability test here first (for the first two parts) and the resource augmentation at the end.

Cynthia A. Phillips, Clifford Stein, Eric Torng, Joel Wein: Optimal Time-Critical Scheduling via Resource Augmentation. STOC 1997: 140-149

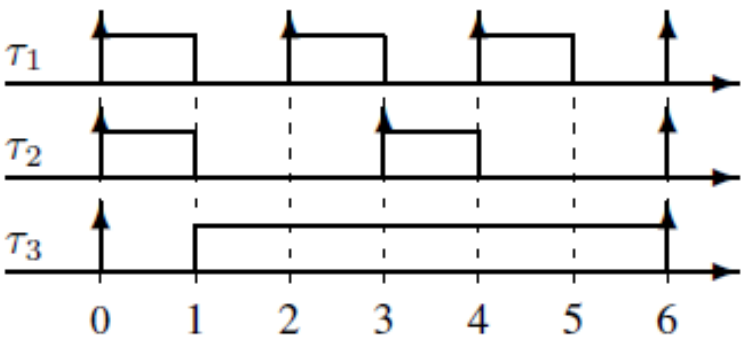
Articles for This Module

- Sanjoy K. Baruah: Techniques for Multiprocessor Global Schedulability Analysis. RTSS 2007: 119-128 (*First part*)
- Nan Guan, Martin Stigge, Wang Yi, Ge Yu: New Response Time Bounds for Fixed Priority Multiprocessor Scheduling. IEEE Real-Time Systems Symposium 2009: 387-397 (*Second part*)
- Vincenzo Bonifaci, Alberto Marchetti-Spaccamela, Sebastian Stiller, Andreas Wiese: Feasibility Analysis in the Sporadic DAG Task Model. ECRTS 2013: 225-233 (*Third part*)
  - Vincenzo Bonifaci, Alberto Marchetti-Spaccamela, Sebastian Stiller: A Constant-Approximate Feasibility Test for Multiprocessor Real-Time Scheduling. ESA 2008: 210-221

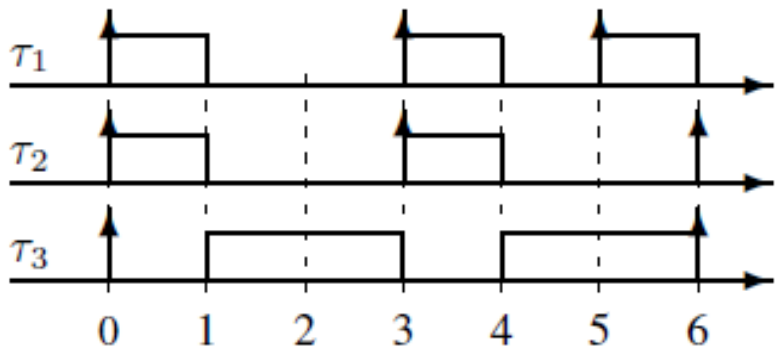
We will mainly focus on task sets with constrained deadlines.

Critical Instants?

- The analysis for uniprocessor scheduling is based on the gold critical instant theorem.
- Synchronous release of higher-priority tasks and as early as possible for the following jobs do not lead to the critical instant for global multiprocessor scheduling
  - Suppose that there two identical processors and 3 tasks: (C_i, D_i, T_i) are $\tau_1 = (1, 2, 2), \tau_2 = (1, 3, 3), \tau_3 = (5, 6, 6)$

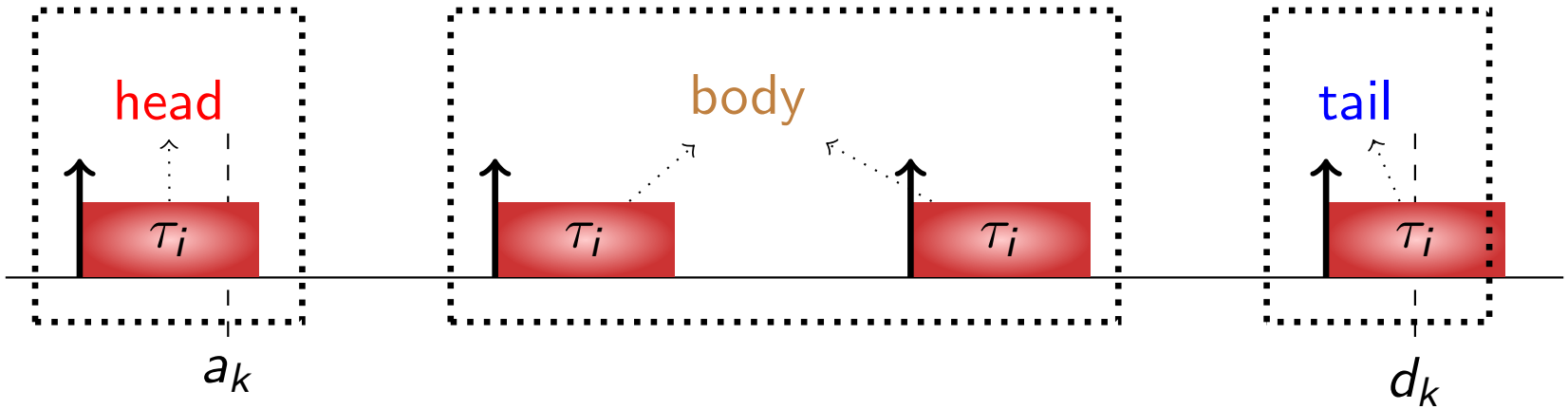


Feasible for τ_3.



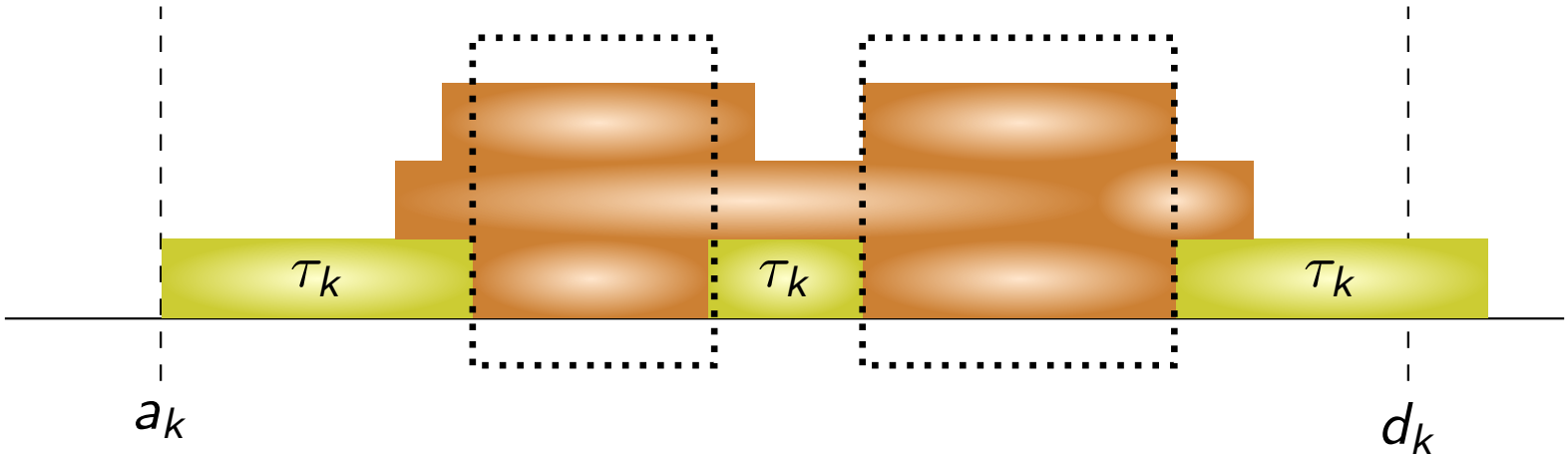
Infeasible for τ_3.

Identifying Interference



- Problem window (interval) is defined in $[a_k, d_k)$.
- The jobs of task τ_i in the problem window can be categorized into three types:
  - Head job (at most one): some computation demand is *carried in* to the problem window for a job arrival before a_k.
  - Body jobs: the computation demand has to be done in the problem window.
  - Tail job (at most one): some computation demand can be *carried out* from the problem window.

Necessary Condition for Deadline Misses



- If τ_k misses the deadline at d_k, there must be at least $D_k - C_k$ units of time in which all M processors are executing other higher-priority jobs.
- Definition: *demand* in a time interval with length Δ is the total amount of computation that needs to be completed within the interval.
- Definition: *load* $W(\Delta)$ is the demand with interval length Δ divided by Δ.
- If τ_k misses its deadline at time d_k, then

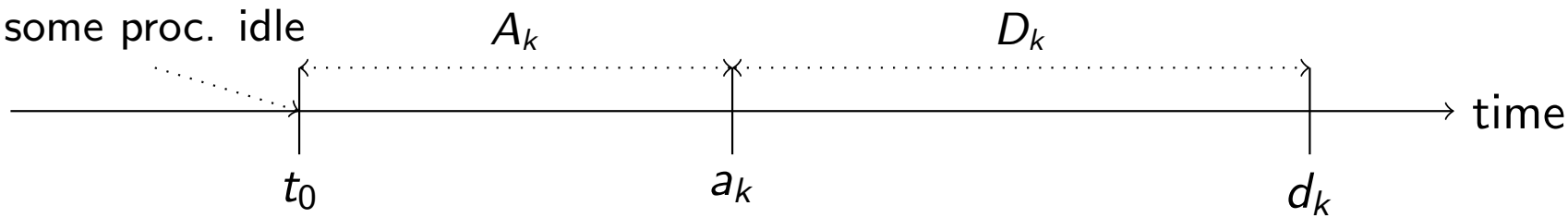
$$W(D_k) > M(1 - \frac{C_k}{D_k}) + \frac{C_k}{D_k}.$$

Outline

Baruah's Approach

For contrapositive, assume that a job of task τ_k misses its absolute deadline at time d_k with release time a_k.

- Bound the carry-in computation demand more precisely.
- Let t_0 be the earliest time instant such that the system executes jobs on M processors from t_0 to a_k.
  - The ready queue before t_0 is with less than M jobs.
  - The ready queue has at least M jobs in time interval $[t_0, a_k)$.
- Let $\mathcal{I}$ be the set of intervals in $[t_0, d_k)$, in which all the M processors are executing. By considering the worst cases, the job of task τ_k arriving at time a_k is not executed at all in $\mathcal{I}$.
- Let A_k be $a_k - t_0$.



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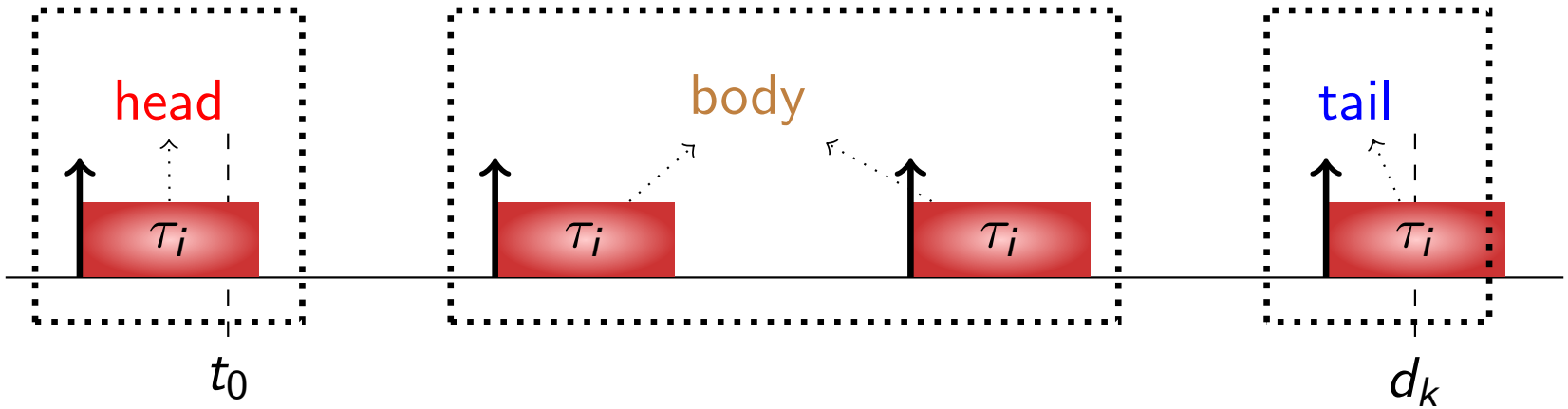

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Necessary Condition for Deadline Misses

- Let $w_i(\mathcal{I})$ be the demand executed in the set $\mathcal{I}$ of time intervals for task τ_i. The necessary condition for τ_k to miss its deadline is

$$\sum_{i=1}^{N} w_i(\mathcal{I}) > M(A_k + D_k - C_k).$$

- Let's consider two types of interferences in $w_i(\mathcal{I})$.
  - Type 1: tasks that are not executing at time t_0. There will be no carry-in demand at time t_0.
  - Type 2: tasks that are executing at time t_0. There might be carry-in demand at time t_0.

Interference Type 1: No Carry-In at Time t_0

- Case 1: $i \neq k$
  - The demand of τ_i to be done in the time intervals in $\mathcal{I}$ is at most
$$\min\left\{dbf(\tau_i, A_k + D_k), A_k + D_k - C_k\right\}.$$
- Case 2: i is k
  - The demand of τ_k to be done in the time intervals in $\mathcal{I}$ is at most
$$\min\left\{dbf(\tau_i, A_k + D_k) - C_k, A_k\right\}.$$
  - Specifically, we need to remove the job that arrives at a_k since its execution is not counted as part of $\mathcal{I}$.

Therefore,

$$w_i^1(\mathcal{I}) =^{\text{def}} \begin{cases} \min\left\{dbf(\tau_i, A_k + D_k), A_k + D_k - C_k\right\} & \text{if } i \neq k \\ \min\left\{dbf(\tau_i, A_k + D_k) - C_k, A_k\right\} & \text{if } i = k. \end{cases}$$

Interference Type 2: With Carry-In at Time t_0

- Case 1: $i \neq k$
  - The demand of τ_i to be done in the time intervals in $\mathcal{I}$ is at most
$$\min\left\{dbf^{\dagger}(\tau_i, A_k + D_k), A_k + D_k - C_k\right\},$$
where $dbf^{\dagger}(\tau_i, \delta)$ is $\left\lfloor \frac{\delta}{T_i} \right\rfloor C_i + \min\{C_i, \delta \mod T_i\}$.
- Case 2: i is k
  - The demand of τ_k to be done in the time intervals in $\mathcal{I}$ is at most
$$\min\left\{dbf^{\dagger}(\tau_i, A_k + D_k) - C_k, A_k\right\}.$$

Therefore,

$$w_i^2(\mathcal{I}) =^{\text{def}} \begin{cases} \min\left\{dbf^{\dagger}(\tau_i, A_k + D_k), A_k + D_k - C_k\right\} & \text{if } i \neq k \\ \min\left\{dbf^{\dagger}(\tau_i, A_k + D_k) - C_k, A_k\right\} & \text{if } i = k. \end{cases}$$

Putting Together

- Let $w_i^{diff}(\mathcal{I})$ be $w_i^2(\mathcal{I}) - w_i^1(\mathcal{I})$.
- The necessary condition for τ_k to miss its deadline becomes

$$\sum_{i=1}^{N} w_i^1(\mathcal{I}) + \sum_{M-1 \text{ largest}} w_i^{diff}(\mathcal{I}) > M(A_k + D_k - C_k).$$

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$$\sum_{i=1}^{N} w_i^1(\mathcal{I}) + \sum_{M-1 \text{ largest}} w_i^{diff}(\mathcal{I}) > M(A_k + D_k - C_k).$$

Theorem

A task set is schedulable under Global-EDF if for every task τ_k and for all $A_k \geq 0$

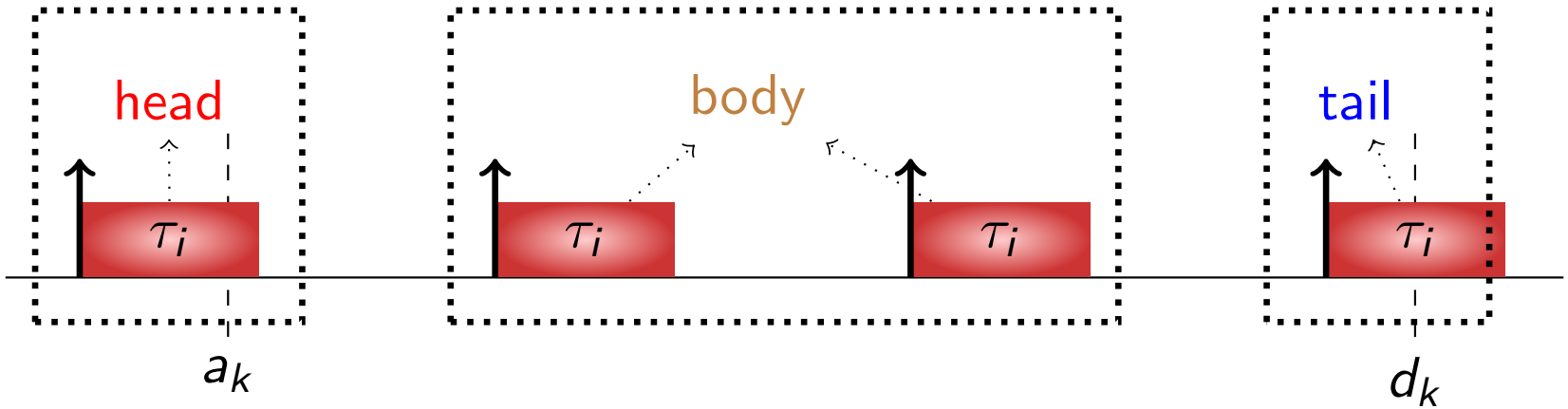
$$\sum_{i=1}^{N} w_i^1(\mathcal{I}) + \sum_{M-1 \text{ largest}} w_i^{diff}(\mathcal{I}) \leq M(A_k + D_k - C_k).$$

Outline

Strategy

- We focus on Global RM in this part. This basically implies that $D_i = T_i$ for every task τ_i. Some of the strategies can be applied to Global DM.
- We are looking for the necessary condition such that a deadline miss happens.
- Suppose that Global scheduling fails by missing the deadline d_k of task τ_k, which is the first instant with deadline missing.
- The job with the earliest deadline miss arrives at time a_k, in which $T_k = D_k = d_k - a_k$.

Identifying Interference



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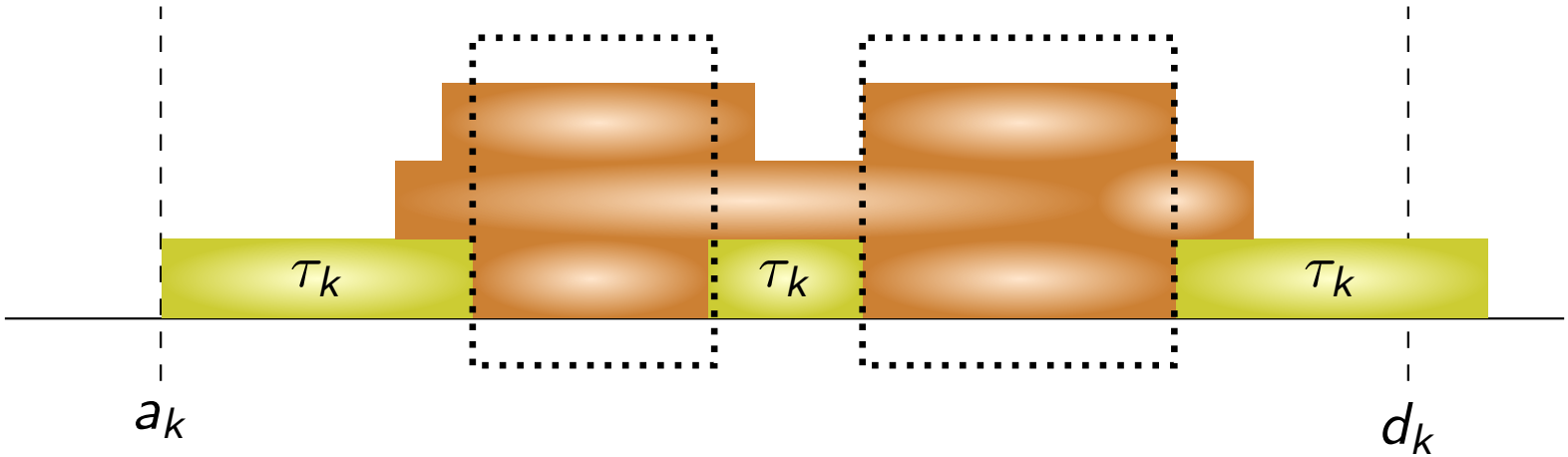
Revisit: Necessary Condition for Deadline Misses



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Bound Carry-In Interference

Theodore P. Baker: Multiprocessor EDF and Deadline Monotonic Schedulability Analysis. RTSS 2003: 120-129 (earlier results)

- Baker's approach tries to bound the carry-in interference by extending the busy-interval to the left hand side while satisfying some load condition.
- This step is called *downward extension of an interval* for global RM. For your reference, the procedures are included in the appendix.
- Here, I am presenting a very simple strategy to analyze the schedulability for global RM.
- This is based on the schedulability analysis we did earlier in the utilization bound analysis for global RM.

A Pessimistic Sufficient Test for Global RM

For all $0 < t \leq T_k$

$$W_k(t) = \sum_{i=1}^{k-1} \left(\left\lceil \frac{t}{T_i} \right\rceil - 1 \right) C_i + 2C_i.$$

This implies that we just greedily take a head job immediately. Clearly, lower-priority jobs have no effect for the unschedulability or schedulability.

Theorem

A system $\mathcal{T}$ of periodic, independent, preemptable tasks is schedulable by Global-RM on M processors if

$$\forall \tau_k \in \mathcal{T} \ \exists t \text{ with } 0 < t \leq T_k \ \textit{ and } \ C_k + \frac{W_k(t)}{M} \leq t$$

holds. This condition is NOT a necessary condition.

Recall k-Point Effective Schedulability Test: k^2U

Suppose that $\{t_1, t_2, \ldots t_k\}$ are given.

Definition

A k-point effective schedulability test is a sufficient test by verifying the existence of $t_j \in \{t_1, t_2, \ldots t_k\}$ with $t_1 \leq t_2 \leq \cdots \leq t_k$ such that

$$C_k + \sum_{i=1}^{k-1} \alpha_i t_i U_i + \sum_{i=1}^{j-1} \beta_i t_i U_i \leq t_j, \tag{1}$$

where $C_k > 0$, $\alpha_i > 0$, $U_i > 0$, and $\beta_i > 0$ are dependent upon the setting of the task models and task τ_i.

Lemma

[Lemma 1] For a given k-point effective schedulability test of a scheduling algorithm, in which $0 < \alpha_i \leq \alpha \neq \infty$, *and* $0 < \beta_i \leq \beta \neq \infty$ *for any* $i = 1, 2, \ldots, k-1$, $0 < t_k \neq \infty$, *task* τ_k *is schedulable by the scheduling algorithm if the following condition holds*

$$\frac{C_k}{t_k} \leq \frac{\frac{\alpha}{\beta} + 1}{\prod_{j=1}^{k-1}(\beta U_j + 1)} - \frac{\alpha}{\beta}. \tag{2}$$

Constrained-Deadline: Schedulability Test for TDA

This is basically very similar to TDA with a minor difference by dividing the higher-priority workload by M. Testing the schedulability condition of task τ_k can be done by using the same strategy used in the k^2U framework.
A simple exercise will lead you to

- $0 < \alpha_i \leq \frac{2}{M}$ and $0 < \beta_i \leq \frac{1}{M}$ for $i = 1, 2, \ldots, k-1$ when testing task τ_k.

Hyperbolic Bound

The task set is schedulable under Global RM if

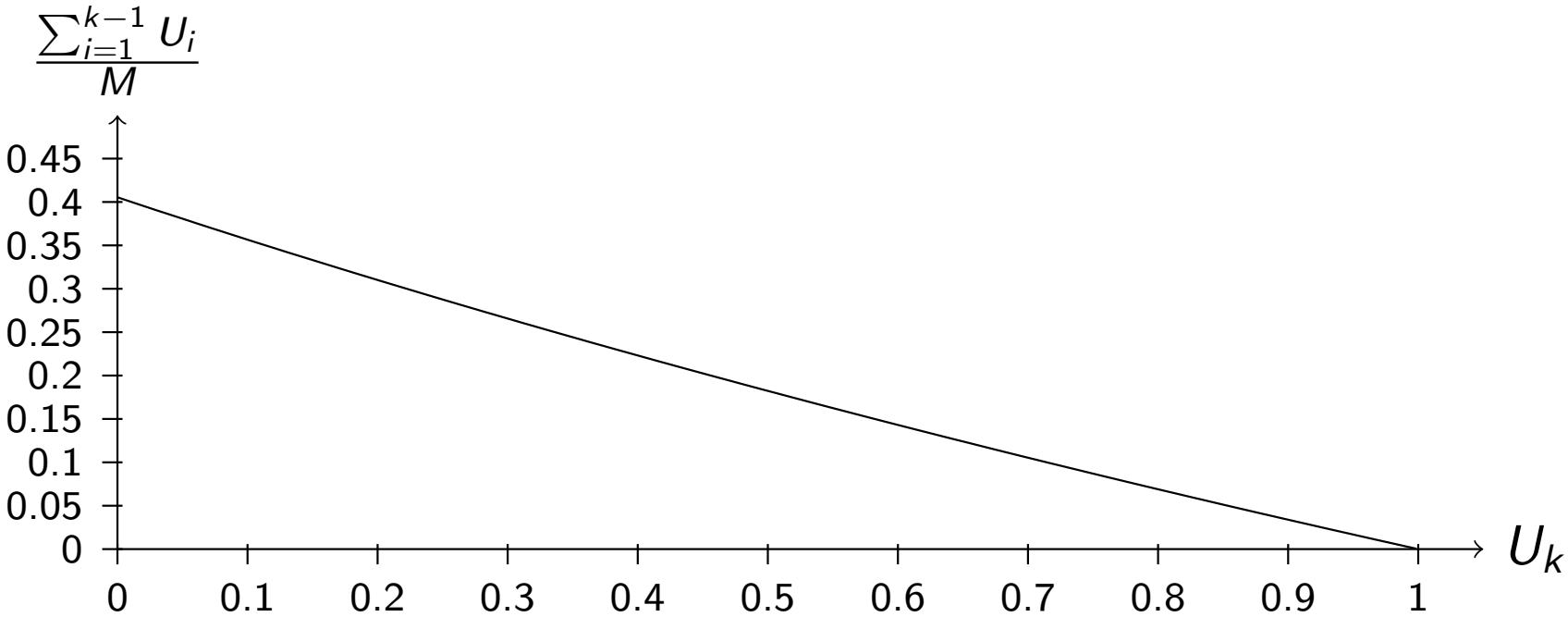
$$\forall k, \qquad (2 + U_k) \prod_{i=1}^{k-1} (U_i/M + 1) \leq 3.$$

Hyperbolic Bound

The task set is schedulable under Global RM if

$$\forall k, \qquad (2 + U_k)\,\Pi_{i=1}^{k-1}(U_i/M + 1) \leq 3.$$

The following figure is the hyperbolic bound for the extreme case when k goes to ∞, in which $(2 + U_k)e^{\frac{\sum_{i=1}^{k-1} U_i}{M}} \leq 3$



Capacity Augmentation Bound

Given a task set $\mathcal{T}$ with total utilization of $U_{\sum}$, a scheduling algorithm $\mathcal{A}$ with capacity augmentation bound b can always schedule this task set on M processors of speed b as long as $\mathcal{T}$ satisfies the following conditions:

$$\text{Utilization does not exceed total cores, } \sum_{\tau_i \in \mathcal{T}} U_i \leq M \tag{3}$$

$$\text{For each task } \tau_i \in \mathcal{T}, \text{the utilization } U_i \leq 1 \tag{4}$$

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This means that the algorithm guarantees the schedulability if the following conditions are satisfied:

$$\text{Utilization does not exceed total cores, } \sum_{\tau_i \in \mathcal{T}} U_i \leq \frac{M}{b} \tag{5}$$

$$\text{For each task } \tau_i \in \mathcal{T}, \text{the utilization } U_i \leq \frac{1}{b} \tag{6}$$

Capacity Augmentation Bound of Global RM

The task set is schedulable under Global RM if

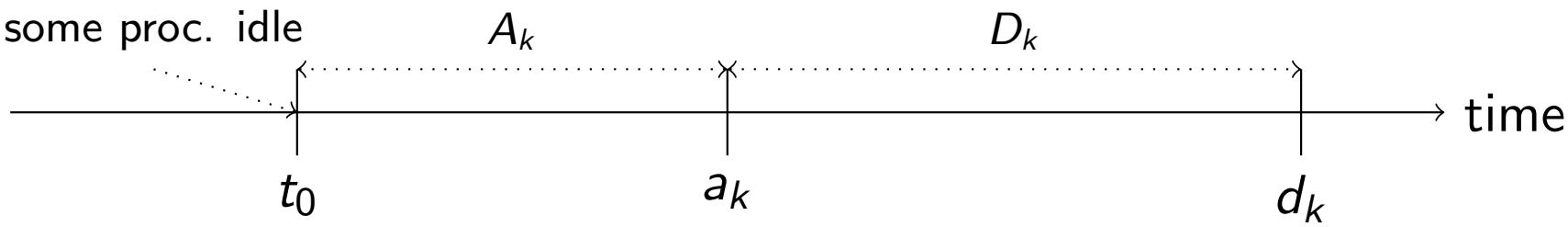
$$\forall k, (2 + U_k) \Pi_{i=1}^{k-1} (U_i/M + 1) \leq 3. \tag{7}$$

$$\Rightarrow \left(2 + \frac{1}{b}\right) \left(\frac{1}{(k-1)b} + 1\right)^{k-1} \leq 3. \tag{8}$$

$$\Rightarrow \left(2 + \frac{1}{b}\right) e^{1/b} \leq 3. \tag{9}$$

Again, we use the worst cases by setting all the tasks with the same utilization as we did in the analysis for uniprocessor systems. This concludes that $b \geq 3.6215$ enforces the above inequality.

Guan's Extension



- Baruah's analysis for Global EDF in fact also works with Global RM for constrained-deadline task systems.
- In the time interval from t_0 to d_k, we only have to consider $M - 1$ tasks with carry-in jobs.

Bounded Carry-In

We can define two different time-demand functions, depending on whether task τ_i is with a carry-in job or not:

$$w_i^2(t) = \begin{cases} C_i & 0 < t < C_i \\ C_i + \left\lceil \frac{t - C_i}{T_i} \right\rceil C_i & otherwise, \end{cases} \tag{10}$$

and

$$w_i^1(t) = \left\lceil \frac{t}{T_i} \right\rceil C_i. \tag{11}$$

We can further over-approximate $w_i^2(t)$, since $w_i^2(t) \leq w_i^1(t) + C_i$. Therefore, a sufficient schedulability test for testing task τ_k with $k > M$ for global RM is to verify whether

$$\exists 0 < t \leq T_k, C_k + \frac{(\sum_{\tau_i \in \mathbf{T}'} C_i) + (\sum_{i=1}^{k-1} w_i^1(t))}{M} \leq t, \tag{12}$$

for all $\mathbf{T}' \subseteq hp(\tau_k)$ with $|\mathbf{T}'| = M - 1$.

Adopting k^2U

There are two ways to use k^2U.

- Case 1: we consider that C_i for task τ_i is known.
  - We simply have to put the $M-1$ higher-priority tasks with the largest execution times into $\mathbf{T}'$.
  - This can be imagined as if we increase the execution time of task τ_k from C_k to $C'_k = C_k + \frac{\sum_{\tau_i \in \mathbf{T}'} C_i}{M}$.
  - Therefore, we still have $0 < \alpha_i \leq \frac{1}{M}$ and $0 < \beta_i \leq \frac{1}{M}$ for $i = 1, 2, \ldots, k-1$
- Case 2: only the task utilizations are given.
  - We need to figure out $\mathbf{T}'$
  - For a higher-priority task τ_i in $\mathbf{T}'$, its α_i is upper-bounded by $\frac{2}{M}$
  - For a higer-priority task τ_i not in $\mathbf{T}'$, its α_i is upper-bounded by $\frac{1}{M}$
  - This is a more complicated case. I am not going to discuss about this.

Adopting k^2U: Case 1

Theorem

Task τ_k in a sporadic implicit-deadline task system is schedulable by global RM on M processors if

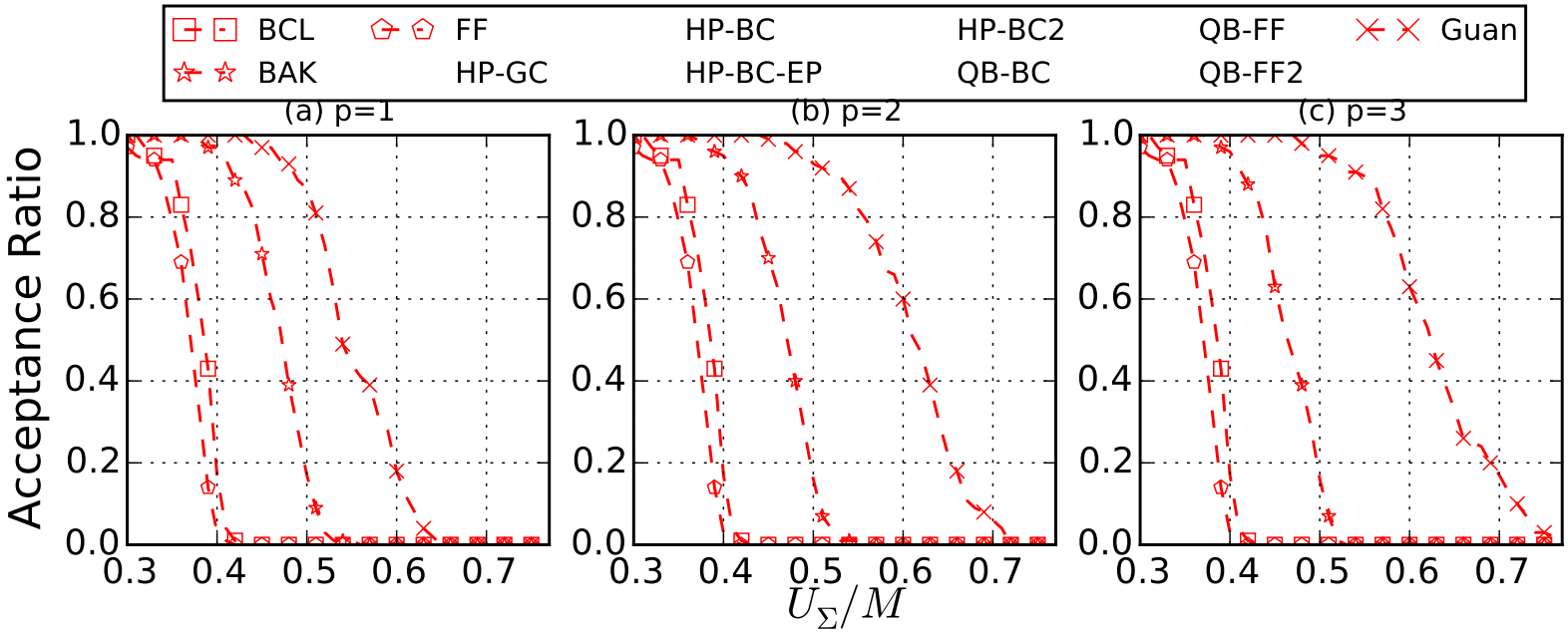
$$\left(\frac{C'_k}{T_k}+1\right)\prod_{i=1}^{k-1}\left(\frac{U_i}{M}+1\right)\leq 2, \tag{13}$$

or

$$\sum_{i=1}^{k-1}\frac{U_i}{M}\leq \ln\left(\frac{2}{\frac{C'_k}{T_k}+2}\right), \tag{14}$$

where $C'_k = C_k + \frac{\sum_{\tau_i\in\mathbf{T}'} C_i}{M}$.

A Brief Look of the Evaluation Results



Chen, Huang, Liu, 2015

Red curves: existing results

- BCL: Bertonga et al. 2005; BAK: Baker 2003; FF: Baruah et al. 2010; Guan: Guan et al. 2009

Blue curves: results by adopting k^2U

Black curves: results by adopting k^2Q

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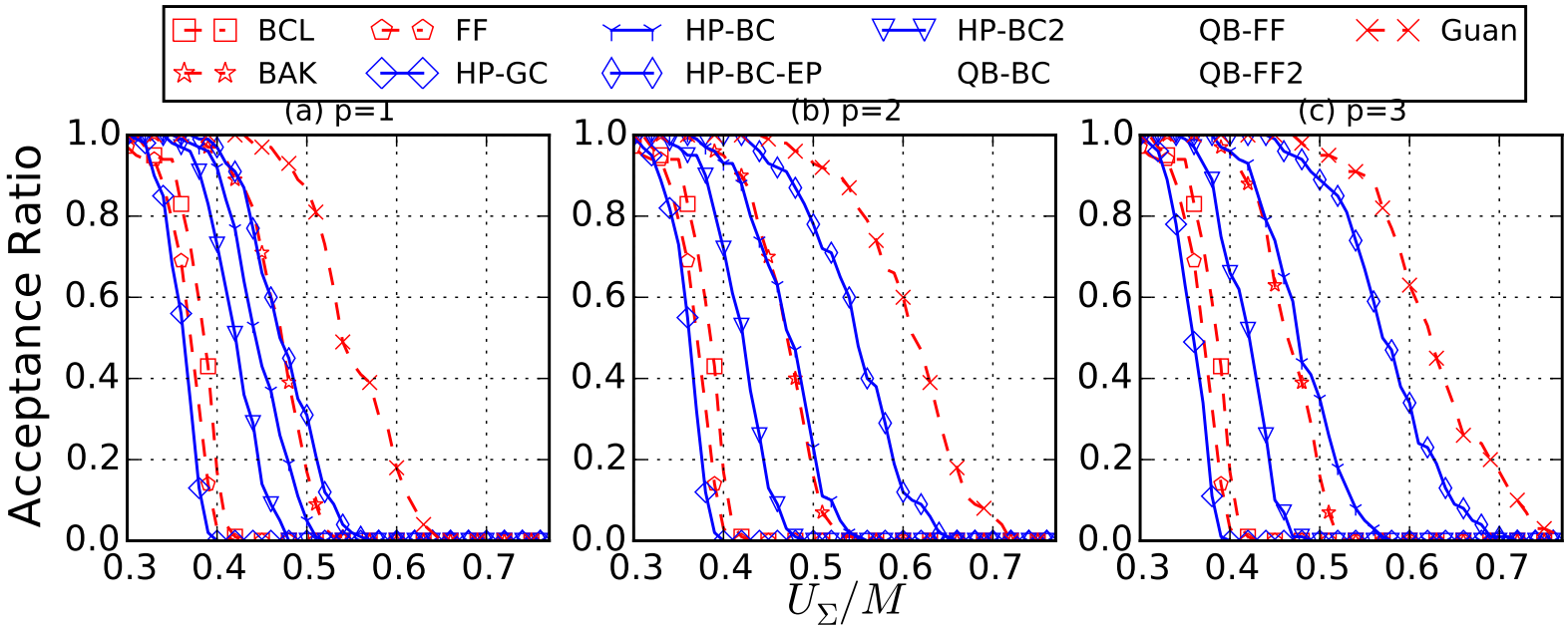

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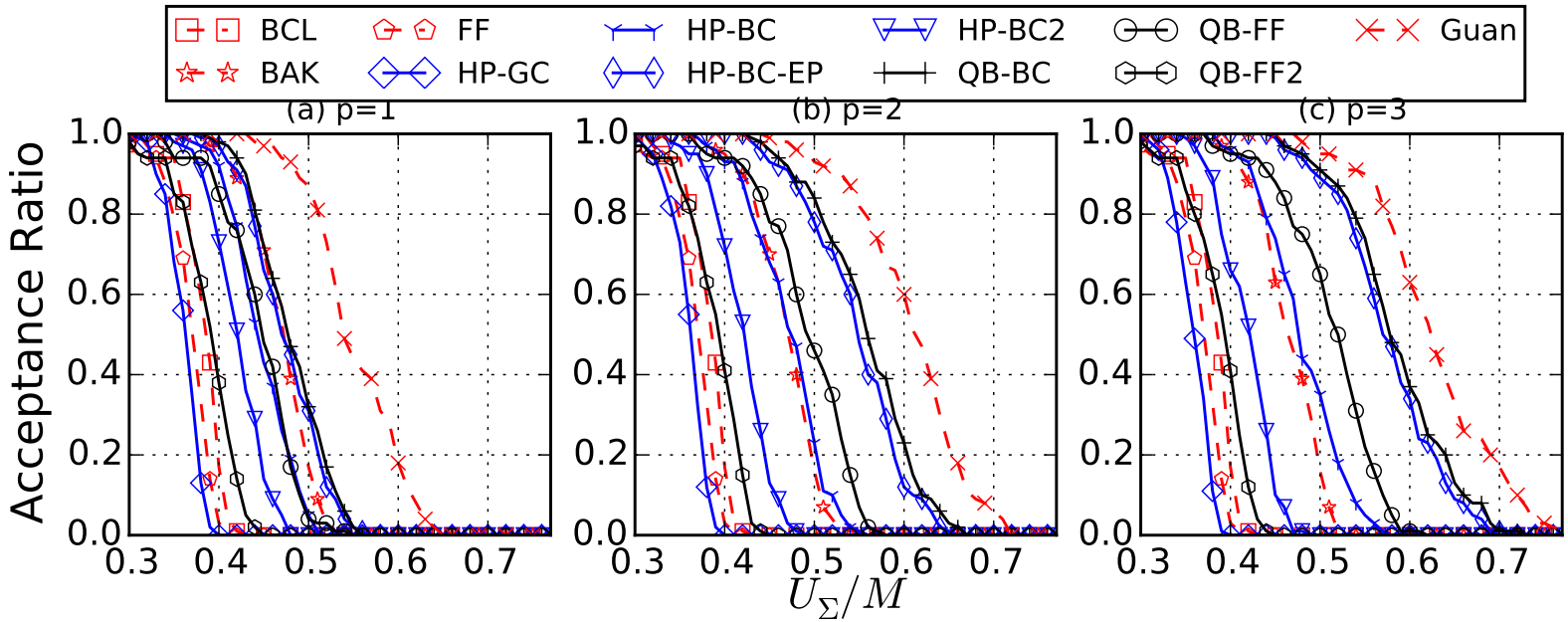
Black curves: results by adopting k^2Q

A Brief Look of the Evaluation Results



Chen, Huang, Liu, 2015

Red curves: existing results

- BCL: Bertonga et al. 2005; BAK: Baker 2003; FF: Baruah et al. 2010; Guan: Guan et al. 2009

Blue curves: results by adopting k^2U

Black curves: results by adopting k^2Q

Summary of Existing Results

Regarding to speedup factors

	implicit deadlines	constrained deadlines	arbitrary deadlines
Global EDF	$2 - \frac{1}{M}$ (Bonifaci et al. 2008)		
Global DM	$3 - \frac{1}{M}$ (Bertogna et al. 2005) $\frac{3+\sqrt{7}}{2} \approx 2.823$ (Chen et al. 2015, k^2Q)	$3 - \frac{1}{M}$ (Baruah et al. 2010) 3 (Chen et al. 2015, k^2Q)	$4 - \frac{1}{M}$ (Baruah/Fisher 2008)

Remarks on Global Scheduling

- pfair: Optimal for implicit-deadline task systems, but with very high overhead. Not introduced in the lecture.
- Global EDF/RM: lower online scheduling overhead, compared to pfair, but not optimal.
- A tradeoff: less management overhead (less task migrations) without losing the optimality.
  - Paul Regnier, George Lima, Ernesto Massa, Greg Levin, Scott A. Brandt: RUN: Optimal Multiprocessor Real-Time Scheduling via Reduction to Uniprocessor. RTSS 2011: 104-115

Outline

Normal Collection of Jobs

A job collection $\mathcal{J}$ is a set of jobs that are revealed online over time:

- a job $j \in \mathcal{J}$ becomes known upon the release date of j
- Each job $j \in \mathcal{J}$ is characterized by its arrival time r_j, absolute deadline d_j, and an unknown execution time c_j.

Note that the actual execution time c_j of a job is discovered by the scheduler only after the job signals completion.

Optimal Schedule for $\mathcal{J}$

Given $\mathcal{J}$, suppose that infinitely many (or, say, $|\mathcal{J}|$) processors of unit speed were available.

Optimal Schedule for $\mathcal{J}$

Given $\mathcal{J}$, suppose that infinitely many (or, say, $|\mathcal{J}|$) processors of unit speed were available.

Then, the following scheduling algorithm S_∞ is optimal:

- just allocate one processor to each job and schedule each job as early as possible.

Schedulability for EDF

Theorem

Consider a normal collection $\mathcal{J}$ of jobs and let $\alpha \geq 1$. Then at least one of the following conditions holds:

1. all jobs in $\mathcal{J}$ are completed within their deadline under EDF on M processors of speed α, or
2. $\mathcal{J}$ is infeasible under S_∞, or
3. there is an interval I such that any feasible schedule for $\mathcal{J}$ must finish more than $(\alpha M - M + 1) \cdot |I|$ units of work within I.

Proof

- The details are omitted, please refer to Bonifaci et al. in ECRTS 2013 (Lemma 3 in Page 228).

Speedup for Normal Collection of Jobs

Theorem

Any normal collection of jobs that is feasible on M processors of unit speed is EDF-schedulable on M processors of speed $2 - 1/M$.

Proof

The feasibility on M processors of unit speed implies that the demand at any interval I is at most $M \cdot |I|$. By setting α to $2 - \frac{1}{M}$, for any interval I, we have

$$(\alpha M - M + 1) \cdot |I| = M \cdot |I|.$$

Hence, this implies that EDF finishes all jobs by their respective deadline at speed $2 - \frac{1}{M}$.

Putting Together

Theorem

If $\forall t > 0$, we have $dbf(\tau_i, t) \leq t$ for every task τ_i and $\sum_{i=1}^{N} dbf(\tau_i, t) \leq M \cdot t$, then this task set with N tasks is EDF-schedulable on M processors of speed $2 - 1/M$.

Theorem

If $\forall t > 0$, we have $dbf(\tau_i, t) \leq \frac{t}{2-1/M}$ for every task τ_i and $\sum_{i=1}^{N} dbf(\tau_i, t) \leq M \cdot \frac{t}{2-1/M}$, then this task set with N tasks is EDF-schedulable on M processors.

This analysis also works for arbitrary deadlines.

Outline

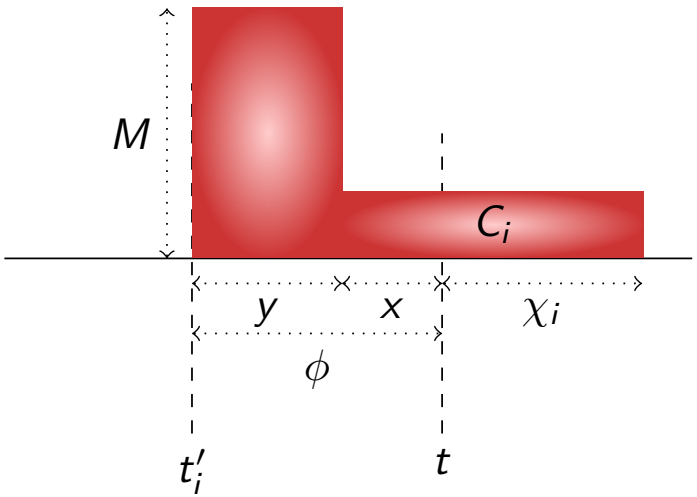
Bound Carry-In Interference

Let t_i' be the last release time of task τ_i before $t \equiv^{\text{def}} a_k$ and ϕ be $t - t_i'$. Suppose that y is the sum of the lengths of all the intervals in $[t_i', t)$, where all M processors are executing jobs that preempt τ_i, then

- If the carry-in χ_i of task τ_i is non-zero, we have

$$\chi_i = C_i - (\phi - y).$$

- The load in interval $[t_i', t)$ is at least $(M-1)\frac{(\phi - C_i + \chi_i)}{\phi} + 1$.



$$y = \phi - (C_i - \chi_i).$$

$$\frac{My + (\phi - y)}{\phi} = (M-1)\frac{y}{\phi} + 1.$$

Busy Intervals

- An interval is said λ-busy if its load is at least $M(1-\lambda)+\lambda$.
- A downward extension of an interval is to expand the starting point of the interval with the original ending point.
- A maximum λ-busy downward extension is the maximum one that can not be extended further to maintain the λ-busy property.
- Example:
  - $[a_k, a_k + D_k)$ is a problem window and is a $\frac{C_k}{D_k}$-busy interval.
- It can be proved that any problem interval for task τ_k has a unique maximal λ-busy downward extension for $\lambda = \frac{C_k}{D_k}$.
  - Let $[t, t+\Delta)$ be the maximal λ-busy downward extension (Note that, therefore, $\Delta \geq D_k$).
  - *t is no longer the arrival time, but $t+\Delta$ is still the deadline.*
- Suppose that the head job of task τ_i $(i \neq k)$ arrives at time $t-\phi$. If $\phi \geq D_i$, there is no carry-in computation of τ_i; otherwise, the carry-in at time t is at most $C_i - \lambda\phi$.

Body and Tail Jobs

- Let n_i be the maximum number of jobs of task τ_i released as the body and tail jobs (that must be completed) in time interval $[t, t + \Delta)$.
- Suppose that $t_i^\dagger$ is the release time of the tail job.
- Let σ_i be $t + \Delta - t_i^\dagger$.

Theorem

Let $n_i = \left\lfloor \frac{\Delta - D_i}{T_i} \right\rfloor + 1$. The EDF demand w_i in the busy window $[t, t + \Delta)$ (the maximal λ-busy downward extension) resulting from τ_i is at most

$$n_i C_i + \max\{0, C_i - \phi\lambda\}.$$

Upper Bound on EDF Load

Theorem

The EDF load $w_i(\Delta)$ in the busy window $[t, t+\Delta)$ (the maximal λ-busy downward extension) for any $\Delta \geq D_k$ is at most

$$\beta_i = \begin{cases} \frac{C_i}{T_i}(1 + \frac{T_i - D_i}{D_k}) & \text{if } \lambda \geq \frac{C_i}{T_i} \\ \frac{C_i}{T_i}(1 + \frac{T_i - D_i}{D_k}) + \frac{C_i - \lambda T_i}{D_k} & \text{if } \lambda < \frac{C_i}{T_i} \end{cases}$$

Proof

- Let $f_i(\Delta)$ be $\frac{n_i C_i + \max\{0, C_i - \phi\lambda\}}{\Delta}$.
- We have known that $w_i(\Delta) \leq f_i(\Delta)$, and would like to prove $f_i(\Delta) \leq \beta_i$

Proof Case 1: $\max\{0, C_i - \phi\lambda\}$ Is 0

- $C_i - \phi\lambda \leq 0 \Rightarrow \lambda \geq \frac{C_i}{\phi}$.
- By definition $\phi < T_i$, and we know $\lambda \geq \frac{C_i}{T_i}$.
- Therefore, we have

$$
\begin{aligned}
f_i(\Delta) = \frac{n_i C_i}{\Delta} &= \frac{(\left\lfloor \frac{\Delta - D_i}{T_i} \right\rfloor + 1) C_i}{\Delta} \leq \frac{\frac{\Delta - D_i + T_i}{T_i} C_i}{\Delta} \\
&= \frac{C_i}{T_i}(1 + \frac{T_i - D_i}{\Delta}) \leq \frac{C_i}{T_i}(1 + \frac{T_i - D_i}{D_k}),
\end{aligned}
$$

where the last inequality comes from the fact $\Delta \geq D_k$.

Proof Case 2: $\max\{0, C_i - \phi\lambda\} > 0$

- $C_i - \phi\lambda > 0$.
- By taking $\phi + \Delta = n_i T_i + D_i$ (i.e., σ_i is D_i) to re-formulate $f_i(\Delta)$, $f_i(\Delta) = \frac{n_i(C_i - \lambda T_i) + C_i - \lambda(D_i - \Delta)}{\Delta}$.
- If $C_i - \lambda T_i > 0$, i.e., $\lambda < \frac{C_i}{T_i}$, we know that ($n_i \leq \frac{\Delta - D_i}{T_i} + 1$)

$$
\begin{aligned}
f_i(\Delta) &\leq \frac{\frac{\Delta - D_i + T_i}{T_i}(C_i - \lambda T_i) + C_i - \lambda(D_i - \Delta)}{\Delta} \\
&\leq \frac{C_i}{T_i}(1 + \frac{T_i - D_i}{\Delta}) + \frac{C_i - \lambda T_i}{\Delta} \\
&\leq \frac{C_i}{T_i}(1 + \frac{T_i - D_i}{D_k}) + \frac{C_i - \lambda T_i}{D_k}
\end{aligned}
$$

- If $C_i - \lambda T_i \leq 0$, i.e., $\lambda \geq \frac{C_i}{T_i}$, we know that ($n_i > \frac{\Delta - D_i}{T_i}$)

$$
f_i(\Delta) < \frac{\frac{\Delta - D_i}{T_i}(C_i - \lambda T_i) + C_i - \lambda(D_i - \Delta)}{\Delta} \leq \frac{C_i}{T_i}(1 + \frac{T_i - D_i}{D_k})
$$

Putting Everything Together

Theorem

A set of periodic tasks $\tau_1, \tau_2, \ldots, \tau_N$ with constrained deadlines is schedulable on M processors by using preemptive Global EDF scheduling if for every task τ_k

$$\sum_{i=1}^{N} \min\{1, \beta_i\} \leq M(1 - \frac{C_k}{D_k}) + \frac{C_k}{D_k}.$$

Theorem

A set of periodic tasks $\tau_1, \tau_2, \ldots, \tau_N$ with implicit deadlines is schedulable on M processors by using preemptive Global EDF scheduling if

$$\sum_{i=1}^{N} \frac{C_i}{T_i} \leq M(1 - \frac{C_k}{T_k}) + \frac{C_k}{T_k},$$

where τ_k is the task with the largest utilization $\frac{C_k}{T_k}$

Upper Bound on DM (Deadline Monotonic) Load

Theorem

The DM load $w_i(\Delta)$ in the busy window $[t, t+\Delta)$ (the maximal λ-busy downward extension) is at most

$$\beta_i = \begin{cases} \frac{C_i}{T_i}(1 + \frac{T_i - \delta_i}{D_k}) & \text{if } \lambda \geq \frac{C_i}{T_i} \\ \frac{C_i}{T_i}(1 + \frac{T_i - \delta_i}{D_k}) + \frac{C_i - \lambda T_i}{D_k} & \text{if } \lambda < \frac{C_i}{T_i} \end{cases},$$

where δ_i is C_i for $i < k$ and δ_k is D_k

Theorem

A set of periodic tasks $\tau_1, \tau_2, \ldots, \tau_N$ with constrained deadlines is schedulable on M processors by using preemptive DM scheduling if for every task τ_k

$$\sum_{i=1}^{k-1} \beta_i \leq M(1 - \frac{C_k}{D_k}).$$