Techniques for Schedulability Analysis in Mode Change Systems under Fixed-Priority Scheduling

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***Abstract*—With the advent of cyber-physical systems, real-time tasks shall be run in different modes over time to react to the change of the physical environment. It is preferable to adopt high expressive models in real-time systems. In the light of simple implementation in kernels, fixed-priority scheduling has been widely adopted in commercial real-time systems. In this work we derive a technique for analyzing schedulability of the system where tasks can undergo mode change under fixed-priority scheduling. We study two types of fixed-priority scheduling in mode change systems: task-level and mode-level fixed-priority scheduling. The propose tests run in polynomial time. The effectiveness of the proposed tests is also shown via extensive simulation results.**

***Keywords*-Real-time scheduling, schedulability analysis, mode changes.**

I. Introduction

In the last decade, accessible networks and sensor devices have become ubiquitous. This gives rise to Cyber-Physical Systems (CPS) in which a system is designed as a network of interacting elements with physical input and output. Such an embedded real-time system continuously monitors and affects the physical environment which also interactively imposes the feedback to the embedded system.

In CPS, the characteristics of a real-time task may be able to change over time, e.g., the computational demand or the resource allocation. Such behavior is referred as *mode changes*. The importance of mode changes for real-time systems has been pointed out in many perspectives, for instance, aircraft control systems, automotive Electronic Control Units (ECU) [10], [20], energy management [22], and server-based systems [2], [16].

In automotive systems, the engine control has different computational demands according to different angular rotations. In each periodic interval, the engine control software calculates the engine speed and position to determine when to fire the next spark signal and evaluates the acceleration/deceleration commands from the driver to adjust the settings of fuel flow [10]. Such a control has the nature of computation mode changes according to the physical environment. Besides, the stringent timing requirement has to be met to inject and to deliver fuel to each cylinder at every revolution.

On the other hand, when the server-based system is used to achieve temporal isolation among applications, the reservation server parameters may need to change from one mode to another [2], [16], [21]. Hence, an additional guarantee is required to ensure feasibility not only in the steady state but also during mode changes.

Related mode change models. In the classical problem of mode changes in hard real-time systems [23], [26], on the request of mode change, all tasks have to switch to their new parameters. Before the new mode of the task system is fully established, no further mode changes are permitted. Several real-time task models have been proposed recently in the literature to analyze the schedulability of adaptive embedded systems.

In this paper, the mode changes are associated to individual tasks. Such mode changes can be found in the generalized multiframe (GMF) model [4], digraph real-time model (DRT) [24], and variable rate-dependent behavior (VRB) task model [7], [8], [10], [13]. The generalized multiframe (GMF) model [4] allows a task to cycle through a static list of job types, each with potentially different WCET bounds and relative deadlines. Stigge et. al. [24] propose a more expressive model, called digraph real-time model (DRT), in which the release structure of different types of jobs are represented by a directed graph. Some approaches on sufficient schedulability tests for DRT model for fixed priority scheduling have been reported [25], whereas the recent study by Guan et al. [12] presents two pseudo-polynomial-time approximations to improve the efficiency by losing some accuracy. In automotive applications, several tasks are linked to rotation (e.g., of the crankshaft, gears, or wheels). Thus their activation rate is proportional to the angular velocity of a specific device. In such a system a common practice is to design a rate-dependent task, called variable rate-dependent behavior (VRB) task model [7], [8], [10], [13]. Typically, at lower rotation some functions that minimize fuel consumption and emissions have to be executed, whereas they are shed at higher rotation speeds to reduce the processor utilization. Table I illustrates an example of a task with four levels of functionality, specified for different speed intervals.

TABLE I: An example of variable-rate behavior task with four types of execution modes dependent on the rotation speed.

rotation (rpm)	functions to be executed
$[0, 2000]$	$f1(); f2(); f3(); f4();$
$(2000, 4000]$	$f1(); f2(); f3();$
$(4000, 6000]$	$f1(); f2();$
$(6000, 8000]$	$f1();$

This work and contribution. In this work we study a real-

time system comprised of n independent, preemptive uniprocessor *multi-mode* tasks $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$ where each task has several execution *modes* to switch during the runtime. The cost of synchronization on mode change within a task is assumed to be subsumed into the worst-case execution time of each mode. A multi-mode task τ_i is denoted by a set of triplets:

$$\tau_i = \{\tau_i^1 = (C_i^1, T_i^1, D_i^1), \\ \tau_i^2 = (C_i^2, T_i^2, D_i^2), ..., \\ \tau_i^{M_i} = (C_i^{M_i}, T_i^{M_i}, D_i^{M_i})\}$$

C_i^m denotes the *worst-case execution time* (WCET) of task τ_i under mode m, T_i^m denotes the *minimum inter-arrival time* of task τ_i under mode m, and D_i^m denotes the *relative deadline*. That is, when a job of mode τ_i^m is released at time t, the next release time of task τ_i is no earlier than $t + T_i^m$, and when a job of mode τ_i^m is released at time t, this job has to be finished no later than its *absolute deadline* at time $t + D_i^m$.

The studied mode change model is a generalization of the sporadic model [17]. The concept of mode change is distinct from that of system-wide operating modes [23], [26]. This model characterizes the system where different *tasks* may progress through their different execution modes independent of each other.

Even the studied mode change model essentially differs from the VRB model where different angular sources drives the inter-arrival time, the mode change model is still applicable when considering only those thresholds that determine which level of functionality should be executed, and in fact the worst-case scenario occurs at the particular value of thresholds. Similar concepts have been presented in [10]. In general, the mode change model can be thought of as a relaxation model of variable rate-dependent behavior (VRB) task model [10], [13] and digraph real-time model (DRT) [24]. From the designer's perspective, the studied mode change model provides an easier way to specify and reason comparing to the more general DRT model.

This paper studies two scheduling algorithms upon multi-mode task systems: fixed-priority *task-level* (FPT) and *mode-level* (FPM). In this work, we derive a technique for analyzing the schedulability in multi-mode systems. Based on the technique, we propose tests that leverage the accuracy and the time complexity for both FPT and FPM scheduling in multi-mode systems.

We summarize our contributions as follows:

1) The proposed test can be efficiently run in polynomial time and is shown to be comparable to the existing state-of-the-art pseudo-polynomial tests for FPT scheduling.
2) Previous tests for determining whether multi-mode systems can be successfully scheduled are only applicable to the system under FPT scheduling. In this work we show that the FPT scheduling can perform rather poorly in the worst case.
3) We prove that a utilization bound of $2-\sqrt{2} \approx 0.5857$ can be guaranteed in implicit-deadline multi-mode systems under rate-monotonic (RM) scheduling, one example of FPM scheduling. Moreover, empirical results show that our proposed tests under RM scheduling is able to accept task sets that are deemed to be schedulable with utilizations of up to 80%.

To the best of our knowledge, this is the first work studying the mode change systems under FPM scheduling.

II. System Properties and Notations

In this work we study a real-time system to execute a set of n independent, preemptive uniprocessor real-time tasks $\tau = \{\tau_1, \tau_2, ..., \tau_n\}$. As presented and defined in Section I, a multi-mode task τ_i with M_i modes is denoted by a set of triplets: $\tau_i = \{\tau_i^1 = (C_i^1, T_i^1, D_i^1), \tau_i^2 = (C_i^2, T_i^2, D_i^2), ..., \tau_i^{M_i} = (C_i^{M_i}, T_i^{M_i}, D_i^{M_i})\}$, to specify the worst-case execution time C_i^m, the minimum inter-arrival time T_i^m, and the relative deadline D_i^m of the corresponding task mode.

Throughout this paper, we restrict ourselves to either *constrained-deadline* ($D_i^m \leq T_i^m$) or *implicit-deadline* ($D_i^m = T_i^m$) multi-mode task systems. For any multi-mode task $\tau_i = \{\tau_i^1 = (C_i^1, T_i^1, D_i^1), \tau_i^2 = (C_i^2, T_i^2, D_i^2), ..., \tau_i^{M_i} = (C_i^{M_i}, T_i^{M_i}, D_i^{M_i})\}$, the utilization U_i^m of mode τ_i^m denotes the ratio C_i^m/T_i^m of its worst-case execution time to the minimum inter-arrival time. We only consider meaningful cases, in which $\sum_{i=1}^{n}(max_{m=1,2,...,M_i} U_i^m) \leq 1$.

We do not impose any constraint on the mode changes between two modes except the temporal distance specified for the minimal inter-arrival time. When a job of task τ_i under the execution mode m is released at time t, the next job of task τ_i cannot be released earlier than $t + T_i^m$, regardless of its next mode. Which mode to be selected as the next mode depends upon the required system properties for reacting to the physical environment or the configuration of the system, which is completely independent from the scheduler. As a result, the difficulty to analyze and schedule such mode change tasks is to precisely quantify and consider the worst-case execution patterns with mode changes.

In this paper we keep our focus on fixed-priority scheduling. We highlight the advantage of fixed-priority scheduling over dynamic scheduling, e.g., *earliest-deadline-first* (EDF), by its light overhead of the implementation. In commercial real-time kernel the explicit support for timing constraints, such as absolute deadlines, is not needed under fixed-priority scheduling.

There are two potential categories of associating the fixed priority level: fixed-priority *task-level* (FPT) and *mode-level* (FPM). In FPM scheduling algorithms, the priority of a mode does not change during runtime; however, different modes of the same task may have different priorities. In contrast, fixed-priority task-level (FPT) algorithms require that all modes of a task have the same priority. It is evident from these definitions that FPM scheduling is a generalization of FTP scheduling. The Rate Monotonic (RM) scheduling algorithm is an example of a FPM scheduling algorithm. The RM scheduling algorithm assigns priority to modes according to the period of the task that generates them: the smaller the period, the greater the priority. Even though FPM allows different priority levels for different modes of a task, all the jobs generated by a mode have the same priority level. In EDF scheduling, the jobs of

a task mode have different priority levels, depending on the absolute deadlines.

The response time of a job in a mode is defined as the completion time of the job minus the release time of the job. The worst-case response time R_k^h of task τ_k for its mode h under a scheduling policy is defined as the longest response time among all released jobs by the mode.

Definition 1 (Schedulability)**.** *A task τ_k is schedulable under a scheduling policy if for every execution mode h of the task, $R_k^h \leq D_k^h$, and a task set is schedulable under a scheduling policy if all of its tasks are schedulable under the scheduling policy.*

We will analyze the schedulability of a task mode τ_k^h under the interference of higher-priority tasks (under FPT) or higher-priority task modes (under FPM). This requires proper definitions of the maximum execution time and the maximum utilization of task τ_i among the modes that are assigned with higher priority than mode τ_k^h. For FPT, as all the modes of a task τ_i are either with higher priority or lower priority than task mode τ_k^h, we define

$$C_i^{max} = max_{\tau_i^m \in \tau_i}(C_i^m)$$

and

$$U_i^{max} = max_{\tau_i^m \in \tau_i}(U_i^m)$$

With the above definition, we denote the total utilization as $U^{sum} = \sum_{i=1}^{n} U_i^{max}$, in which $U^{sum} \leq 1$. Note that the task modes resulting in C_i^{max} and U_i^{max} may be different, we further define the load factor of task τ_i as follows:

$$\beta_i = \frac{C_i^{max}}{U_i^{max}}$$

For FPM, let $hp(\tau_k^h)$ denote the *set* of the modes that are assigned with higher priority than mode τ_k^h. Therefore, the intersect set $hp(\tau_k^h) \cap \tau_i$ consists of the task modes of task τ_i that are assigned with higher priority than $hp(\tau_k^h)$. Similarly, the maximum execution time and the maximum utilization of task τ_j among the modes that are assigned with higher priority than mode τ_k^h are denoted as follows:

$$C_i^{max}(\tau_k^h) = max_{\tau_i^m \in (hp(\tau_k^h) \cap \tau_i)}(C_i^m)$$

and

$$U_i^{max}(\tau_k^h) = max_{\tau_i^m \in (hp(\tau_k^h) \cap \tau_i)}(U_i^m)$$

Similarly, the load factor of task τ_i with respect to mode τ_k^h in FPM is

$$\beta_i(\tau_k^h) = \frac{C_i^{max}(\tau_k^h)}{U_i^{max}(\tau_k^h)}$$

Note that if task τ_i does not have any higher-priority task mode than mode τ_k^h, we can simply remove such a task τ_i in the schedulability analysis of task mode τ_k^h.

III. Problem Statement and Existing Results

The objective of the mode change schedulability analysis is to guarantee that a system is feasible not only in the steady state but also during the mode transition, during which the phenomenon of demand strides may occur and lead to an unfeasible scheduling, even provided that there is no deadline miss in the steady state. The following example will further motivate the difficulty of mode change.

Example Consider a system with one multi-mode task and one sporadic task: $\tau_1 = \{(2,3,3),(4,8,8)\}$ and $\tau_2 = \{(4,12,12)\}$. Both tasks are schedulable by RM without mode transition. However, task τ_2 will miss its deadline at time-instant 12 when task τ_1 switches from mode τ_1^2 to mode τ_1^1 at time-instant 9, as illustrated in Figure 1. □

As a result, it is inevitable that the combinatorial releases have to be taken into account to identify the worst-case scenario. Towards this, we are looking for the *critical instant* for a task, which is defined as the instant at which an execution of that task will have the longest response time [14]. For sporadic tasks with constrained deadlines, it is proved that a critical instant for a task occurs whenever the task is released simultaneously with all higher priority tasks and all the jobs are released as soon as possible [14].

The critical instant for a multi-mode task under FPT scheduling has been presented in Theorem 1 in [10].[1] For completeness, we paraphrase this theorem in the following lemma :

Lemma 1 (Davis et al. [10])**.** *There is a sequence Y of jobs of task τ_i , that τ_i releases the maximum interference $I_i(w)$ in a window $[0, w)$, where*

1) *the offset, from the start of the window, of the first job of τ_i is zero;*
2) *each of the jobs of τ_i released in the window has the minimum period commensurate with its particular execution mode;*
3) *the last job has the largest WCET for any execution mode.*

We notice that unlike for the sporadic task, the critical instant provided in Lemma 1 is only necessary for creating the maximum interference. The *exact* worst-case interference from all tasks has to enumerate all possible release sequences that satisfy the above conditions for all the tasks, but is, however, computationally intractable since the time complexity is $O(W_1 \times W_2 \times ...W_{n-1})$ where W_i is the possible release sequences within the input period and is $\geq M_i$. As an amenable solution, instead of enumerating all possible sequences for all tasks, a method using integer linear programming (ILP) solvers for deriving the maximum demand has been proposed in [10]. Also, a dynamic programming method has been reported in [24] for the DRT model, which is a generalization of mode change model. However, both approaches may lead to an over-approximation of the exact response time, due to the non-concrete traces.

IV. Schedulabiliy Analysis under FPT Scheduling

In this section, we first introduce the concept of the multi-mode demand bound function, and derive a sufficient schedu-

[1] Although the focus in [10] was for the VRB task model, the critical instant theorem in [10] works for the mode change task model in this paper as well. The proof to create the maximum interference is exactly the same.

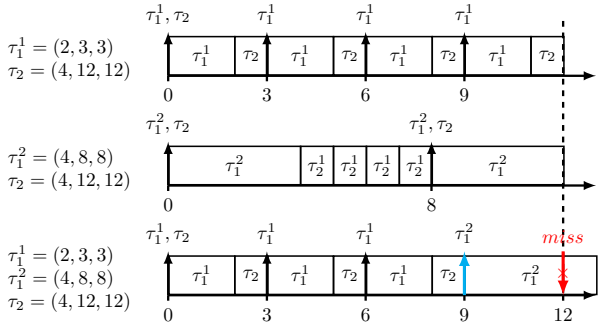


Figure 1: The missed deadline during mode transition

lability test for multi-mode task systems under a given FPT assignment, in Theorem 1,.

By Lemma 1, no matter how the sequence of task τ_i releases prior to the arrival time of the last release, we can observe this deadline miss after replacing the last release mode in $[0, w)$ by the mode with the largest WCET among all modes. Based on this observation, we can decompose the interference from a multi-mode task into two parts: (*i*) the execution time from the last release that has the largest WCET among modes and (*ii*) the total demand prior to the arrival time of the last release. We formally define the last release time and multi-mode demand bound function as follows:

Definition 2 (Last Release Time)**.** *For a given sequence of releases of task τ_i in the interval $[t_0, t_0 + t)$, we define t_i as the time of the last release of task τ_i upon the sequence.*

Definition 3 (Multi-mode demand bound function)**.** *For any interval length of t, the demand bound function $DBF_i(t, \tau_k^h)$ of a multi-mode task τ_i is defined as the maximum cumulative execution requirement by jobs of τ_i that are assigned with higher priority than mode τ_k^h and have both arrive time and next release time within an interval length of t.*

A. Multi-Mode Demand Bound Function

The concept of demand bound function (DBF) has been widely used in real-time schedulability analysis [5]. The conventional *demand bound function* (DBF) [5] bounds the maximum cumulative execution requirement by jobs of *sporadic* task τ_i that both arrive in and have absolute deadlines within any interval of length t. As for the multi-mode demand bound function, the function involves the combinatorial releases.

In fact, by the definition of the multi-mode workload, calculating the multi-mode demand bound function is equivalent to the well-known *unbounded knapsack problem* (UKP) [15]. The *unbounded knapsack problem* is to determine the number of each item to include in a collection of items so that total weight (execution time) of the selected items is less than or equal to a given limit (interval length) (called knapsack) and total value (cumulative executions) of the selected items is maximized.

As a result, the multi-mode demand bound function can be computed in pseudo-polynomial time using *dynamic programming* [15]. Nevertheless, it has been shown in [15] that an upper bound B for UKP is

$$B = \left\lfloor c \frac{p_1}{w_1} \right\rfloor \tag{1}$$

where p_i denotes the *profit* of an item of type i, w_i denotes the *weight* of an item of type i, c is the limit of the knapsack, and the item types are ordered so that

$$\frac{p_1}{w_1} \geq \frac{p_2}{w_2} \geq ... \geq \frac{p_n}{w_n} \tag{2}$$

In the above transformations, $\frac{p_1}{w_1}$ is corresponding to the maximum utilization among the eligible modes. Lemma 2 follows immediately:

Lemma 2. *For any τ_i and a higher-priority mode τ_k^h, for $t > 0$*

$$DBF_i(t, \tau_k^h) \leq t \cdot U_i^{max}(\tau_k^h)$$

Proof: Notice that $\frac{p_1}{w_1} = U_i^{max}(\tau_k^h)$.

$$\begin{aligned} \text{By Eq. (1)} \quad & DBF_i(t, \tau_k^h) \leq \left\lfloor t U_i^{max}(\tau_k^h) \right\rfloor \\ \Rightarrow\ & DBF_i(t, \tau_k^h) \leq t \cdot U_i^{max}(\tau_k^h) \end{aligned}$$

■

It is worth noting that by definition it is possible that the mode belonging to $U_i^{max}(\tau_k^h)$ has a period $> t$ and thus cannot both arrive in and have next release time within interval length of t. Nevertheless, the upper bound still holds. We will use the upper bound on the multi-mode demand function to derive a schedulability technique that requires the *continuity* of the upper bound with the interval length of t. The mode having period larger than t can only be ruled out when the interval of interest is fixed, e.g. under FPT $0 < t \leq D_k^h$. Due to space limitation, we do not explicitly present the detail.

B. Sufficient Test

In this section, we use the concept of the interference decomposition to derive a technique for analyzing schedulability of real-time systems represented using the mode change model.

Consider any legal sequence of jobs of task system τ, on which a deadline miss occurs. Suppose that a mode m of the kth- highest priority task is the one to first miss a deadline and that the mode arrives at time-instant t_a and this deadline miss occurs at time-instant $t_a + D_k^h$.

Without loss of generality, by Lemma 1, we set $t_a = 0$. We first assume that the tasks that are assigned with higher priority than mode τ_k^h are indexed according to the last release time ordering π, upon which the deadline miss occurs.

Definition 4 (Last Release Time Ordering)**.** *Let π be the assignment of a last release time ordering as a bijective function $\pi : \tau \to \{1, 2, \ldots, k-1\}$ to define the last release time ordering of task $\tau_i \in hp(\tau_k^h)$. The ordering of last releases is numbered from 1 to $k-1$ where 1 is the earliest and $k-1$ the latest.*

Lemma 3. *If task mode τ_k^h misses its deadline under FPT upon a last release assignment π, then there exists an assign-*

ment $t_1, t_2, \ldots, t_{k-1} \in [0, D_k^h)$ *of the last release of task* τ_i *such that* $\forall i \in \{1, ..., k-1, k\}$

$$\sum_{\tau_j \in hp(\tau_k^h)} DBF_j(t_j, \tau_k^h) + \sum_{j=1}^{i-1} C_j^{max} + C_k^h > t_i \quad (3)$$

where $t_k \equiv D_k^h$.

Proof: By Lemma 1 and the assumption of the deadline miss of mode τ_k^h, the released pattern described in Lemma 1 will result in a deadline miss for the job released by mode τ_k^h at time $t_a = 0$. Let π be such a last release ordering, and define the last release time t_i of task τ_i for $i = 1, 2, \ldots, k-1$ accordingly with $t_i \leq t_{i+1}$. The executed workload of the job released by mode τ_k^h must be strictly less than C_k^h amount of execution time over $[0, D_k^h]$. We observe the followings:

- At time point t_i for any $i = 1, 2, \ldots, k-1$, the requested higher-priority execution time prior to t_i plus C_k^h is larger than t_i.
- Up to time t_i, a task τ_j with $j = 1, 2, \ldots, i-1$ has requested $DBF_j(t_j, \tau_k^h) + C_j^{max}$ amount of execution time to be executed.
- Up to time t_i, a task τ_j with $j = i, i+1, \ldots, k-1$ has requested $DBF_j(t_i, \tau_k^h)$ amount of execution time to be executed.

The above observation results in $\forall i = 1, 2, \ldots, k$,

$$C_k^h + \sum_{j=1}^{i-1} (DBF_j(t_j, \tau_k^h) + C_j^{max}) + \sum_{j=i}^{k-1} DBF_j(t_i, \tau_k^h) > t_i$$

By the fact that $DBF_j(t, \tau_k^h)$ is monotonically non-decreasing with respect to the interval length t, we know that $DBF_j(t_i, \tau_k^h) \leq DBF_j(t_j, \tau_k^h)$ when $j \geq i$. (Due to the last release ordering π, we have $t_j \geq t_i$ for such cases.) As a result, we reach the conclusion in Eq. (3). ∎

Lemma 4. *If mode* τ_k^h *misses its deadline upon a last release time assignment* π*, it must be either the case that*

$$C_k^h + \sum_{i=1}^{k-1} C_i^{max} > D_k^h \quad (4)$$

or

$$C_k^h > D_k^h - \sum_{i=1}^{k-1} \left(U_i^{max} \cdot \left(D_k^h - \sum_{j=i}^{k-1} C_j^{max} \right) \right) - \sum_{i=1}^{k-1} C_i^{max} \quad (5)$$

Proof: In case of Eq. (4), it is clear that there will be a deadline miss.

We now consider the case where Eq. (4) dose not hold. From Lemma 3, we know that $\forall i \in \{1, ..., k-1, k\}$

$$\sum_{j=1}^{k-1} DBF_j(t_j, \tau_k^h) + \sum_{j=1}^{i-1} C_j^{max} + C_k^h > t_i$$

$$\text{(By Lemma 2)} \Rightarrow \sum_{j=1}^{k-1} U_j^{max} t_j + \sum_{j=1}^{i-1} C_j^{max} + C_k^h > t_i$$

Our objective is to find the minimum C_k^h such that the above constraints always hold. This is equivalent to the following linear programming (LP):

$$\begin{aligned}
\min \quad & C^* \\
\text{s.t.} \quad & \sum_{j=1}^{k-1} U_j^{max} t_j + \sum_{j=1}^{i-1} C_j^{max} + C^* > t_i, && \forall 1 \leq i \leq k-1 && \text{(6a)} \\
& \sum_{j=1}^{k-1} U_j^{max} t_j + \sum_{j=1}^{k-1} C_j^{max} + C^* > D_k^h && && \text{(6b)} \\
& t_{k-1} < D_k^h, && && \text{(6c)} \\
& t_i \leq t_{i+1}, && \forall 1 \leq i \leq k-2 && \text{(6d)} \\
& t_i \geq 0, && \forall 1 \leq i \leq k-1 && \text{(6e)}
\end{aligned}$$

where Eq. (6e), Eq. (6d), and Eq. (6c) come from the definition of t_i. Given an LP, there are three possibilities:

1) The polyhedron is *infeasible*.
2) The objective function can be made arbitrarily large/small, so to speak *unbounded*.
3) There is a finite optimum value.

We now show that this LP has a finite optimum value where there exists an extreme point of the feasible region which is optimal.

Consider the points $t_1 = t_2 = ... = t_{k-1} = 0$ where all constraints are satisfied due to the fact $C_i^{max} > 0$ for all tasks τ_i. The polyhedron is hence feasible. Due to the bounded variables t_i, the objective function is bounded.

In the remaining proof we can solve this LP as the basic solution where an optimal solution must occur at a vertex of the set of feasible solutions and at least k constraints must be *active*[2] at that vector [18].

For the simplicity of the proof, we hereby remove the constraints of Eq. (6c) and (6d) from our problem. The constraints to be deleted may be either binding or unbinding (redundant) on the optimal solution. The deletion of an unbinding constraint can only enlarge the feasible region but will not affect the optimal solution. The deletion of a binding constraint may, however, cause post-optimality problem, but it will not lead to any false-negatives. Note that this deletion, in fact, will not cause such a problem.

Case 1: Eq. (6a) and Eq. (6b)'s activation.

Consider an extreme point where all constraints of Eq. (6a) and Eq. (6b) are active. By subtracting the adjacent constraints in Eq. (6a) and Eq. (6b), we then obtain that $\forall 1 \leq i \leq k-2$, $t_i^* = t_{i+1}^* - C_i^{max}$ and $t_{k-1}^* = D_k^h - C_{k-1}^{max}$ which can be rewritten as the equality

$$t_i^* = D_k^h - \sum_{j=i}^{k-1} C_j^{max}$$

By putting the above equalities to the activation of Eq. (6b), i.e.,$\sum_{j=1}^{k-1} U_j^{max} t_j + \sum_{j=1}^{k-1} C_j^{max} + C^* = D_k^h$, we then obtain

[2]Given a point x in the feasible region, a constraint $g_i(x) \geq 0$ is called *active* at x if $g_i(x) = 0$

that

$$C^* = D_k^h - \sum_{i=1}^{k-1} (U_i^{max} \cdot t_i^*) - \sum_{i=1}^{k-1} C_i^{max}$$

which is equivalent to the RHS of Eq. (5). In addition, due to the unsatisfied Eq. (4), this extreme point is feasible since for every last release t_i

$$t_i^* = D_k^h - \sum_{j=i}^{k-1} C_j^{max} \geq 0$$

In fact, this extreme point is literally the so-called optimum, and none of the others can be less than it.

Case 2: Eq. (6e)'s activation.

Observe that when whichever Eq. (6e) is *active*, its counterpart of Eq. (6a) becomes inactive. To be active for at least k constraints, Eq. (6b) has to be active, i.e. $\sum_{j=1}^{k-1} U_j^{max} t_j + \sum_{j=1}^{k-1} C_j^{max} + C^* = D_k^h$.

Let S denote the *set* of last release times t_i that are active in Eq. (6a). Let $\hat{t}_i^*$ denote the extreme point of t_i for this case. Consider any two last release times $t_r, t_s \in S$. Without loss of generality, we assume $r > s$. Similarly, we know that $\hat{t}_r^* - \hat{t}_s^* = \sum_{j=s}^{r-1} C_j^{max}$. Since Eq. (6b) has to be active, the above equalities can be rewritten as follows:

$$\hat{t}_i^* = \begin{cases} D_k^h - \sum_{j=i}^{k-1} C_j & \text{if } t_i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

which is no more than t_i^* in Case 1. Observe that Eq. (6b) is active, we can conclude that Case 2 is not the optimum with simple logic. Hence this lemma is proved. ∎

Lemma 4 suggests the cases when a deadline miss occurs upon a last release ordering π. Without knowing the exact ordering observed in the schedule, intuitively, one can relax the ordering assumption by examining all the possible permutations. However, this requires to examine all possible $(k-1)!$ permutations. Fortunately, the following lemma shows that $D_k^h - \sum_{i=1}^{k-1} \left(U_i^{max} \cdot \left(D_k^h - \sum_{j=i}^{k-1} C_j^{max} \right) \right) - \sum_{i=1}^{k-1} C_i^{max}$ is minimized for a specific ordering of π.

Lemma 5. *$D_k^h - \sum_{i=1}^{k-1} \left(U_i^{max} \cdot \left(D_k^h - \sum_{j=i}^{k-1} C_j^{max} \right) \right) - \sum_{i=1}^{k-1} C_i^{max}$ is minimized when the last release time ordering π of the $k-1$ higher priority tasks is with a non-increasing order of β_i.*

Proof: Suppose that π^* does not follow a non-increasing order of β_i. That is, there exists ℓ in the ordering π^* such that $\beta_\ell < \beta_{\ell+1}$ for some $\ell = 1, 2, \ldots, k-2$. We can now swap these two tasks, and this results in a new last release time ordering π'. By inspecting the term to be proved, we know that the last release ordering only changes $\sum_{i=1}^{k-1} \left(U_i^{max} \sum_{j=i}^{k-1} C_j^{max} \right)$. Therefore, to prove the lemma, we just have to show that the ordering π' has smaller $\sum_{i=1}^{k-1} \left(U_i^{max} \sum_{j=i}^{k-1} C_j^{max} \right)$ than the ordering π^*. By comparing these two orderings π^* and π', the term $\left(U_i^{max} \sum_{j=i}^{k-1} C_j^{max} \right)$ remains the same in π and π' for any $i \neq \ell$ and $i \neq \ell+1$. Therefore, the difference (the result by using π' minus the result by using π^*) in the term $\sum_{i=1}^{k-1} \left(U_i^{max} \sum_{j=i}^{k-1} C_j^{max} \right)$ is $\left(U_{\ell+1}^{max} \sum_{j=\ell}^{k-1} C_j^{max} + U_\ell^{max} \sum_{j=\ell+1}^{k-1} C_j^{max} \right) - \left(U_\ell^{max} \sum_{j=\ell}^{k-1} C_j^{max} + U_{\ell+1}^{max} \sum_{j=\ell+1}^{k-1} C_j^{max} \right)$, which results in

$$\begin{aligned} U_{\ell+1}^{max} C_\ell^{max} - U_\ell^{max} C_{\ell+1}^{max} &= U_{\ell+1}^{max} U_\ell^{max} \left(\frac{C_\ell^{max}}{U_\ell^{max}} - \frac{C_{\ell+1}^{max}}{U_{\ell+1}^{max}} \right) \\ &= U_{\ell+1}^{max} U_\ell^{max} (\beta_\ell - \beta_{\ell+1}) < 0, \end{aligned}$$

where the last inequality comes from the definition of ℓ.

Therefore, we can keep swapping the last release ordering to reduce $\sum_{i=1}^{k-1} \left(U_i^{max} \sum_{j=i}^{k-1} C_j^{max} \right)$. By adopting the above swapping procedure repeatedly, we reach the conclusion. ∎

Since Lemma 4 is the necessary condition for a deadline miss to occur, equivalently, the negation of Lemma 4 is the sufficient condition for a deadline to be met. We can now conclude the following schedulability test by using Lemma 4 and Lemma 5.

Theorem 1 (QT-FPT)**.** *A constrained-deadline multi-mode task τ_k^h is schedulable if*

$$D_k^h - \sum_{i=1}^{k-1} C_i^{max} - C_k^h \geq 0$$

and

$$C_k^h \leq D_k^h - \sum_{i=1}^{k-1} \left(U_i^{max} \cdot \left(D_k^h - \sum_{j=i}^{k-1} C_j^{max} \right) \right) - \sum_{i=1}^{k-1} C_i^{max}$$

where the higher priority tasks τ_i are indexed by non-increasing β_i.

C. Computational complexity

The higher-priority tasks have to be sorted by non-increasing β_i beforehand, and hence the test runs in time of $O(n \log n)$ for each mode. In addition, the time complexity for determining whether a multi-mode task system is schedulable is $O(n \log n \cdot (M_1 + M_2 + ... + M_n))$ where $(M_1 + M_2 + ... + M_n)$ is for checking all the task modes. Therefore, the proposed test can be computed in polynomial time.

V. Schedulabiliy Analysis under FPM Scheduling

In this section, we derive the schedulability test for multi-mode systems under FPM scheduling by using the similar technique provided in Section IV. Specifically, we study RM scheduling, one example of FPM, and then provide utilization bounds, in Theorem 4 and Theorem 5, for an implicit-deadline multi-mode system.

Several results have been reported on FPT scheduling [10], [19], [25]. We here show that FPT scheduling may perform rather poorly comparing to FPM scheduling.

Example. Consider the set of tasks defined in Table II where task τ_1 is a sporadic task; τ_2 is a multi-mode task; $0 < \epsilon \leq 0.5$. If τ_1 has higher priority than τ_2, mode τ_2^1 cannot meet its deadline. Similarly, if τ_2 has higher priority than τ_1, task τ_1

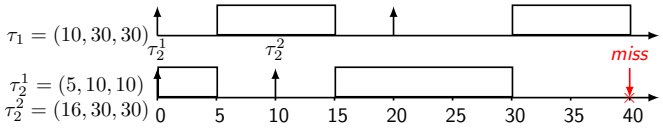


Figure 2: An example of carry-in effect under FPM. Mode τ_1 carries five time-units job, that does not occur with synchronous releases, into the interval of task τ_2^2 due to the preemption from higher-priority mode τ_2^1.

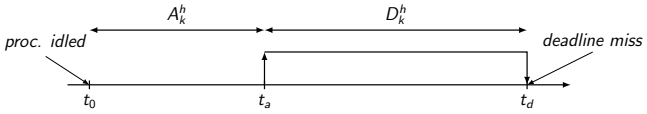


Figure 3: Illustration: a mode of task τ_i arrives at t_a and misses its deadline at time-instant t_d. The latest time-instant prior to t_a when the processor is idle is denoted t_0.

cannot meet its deadline due to the preemption resulting from the mode τ_2^2.

Task	Mode	C_i^m	T_i^m	D_i^m
τ_1	1	1	$1/\epsilon$	$1/\epsilon$
τ_2	1	ϵ	1	1
	2	$1/\epsilon$	$(1/\epsilon)^2$	$(1/\epsilon)^2$

TABLE II: An example of associating fixed priority with the task set

According to the example, when FPT is adopted, a feasible scheduling that will always meet deadlines does not exist, no matter task τ_1 or task τ_2 is assigned to the higher priority for any $\epsilon > 0$. Therefore, the utilization bound for such a case is 0 when ϵ is close to 0. That is, there exist input instances with a very small utilization such that the task set is not schedulable under any FPT assignments.

A naive way to check the schedulability under RM is to pessimistically consider each mode as an individual sporadic task. Subsequently, by inspecting the Liu & Layland bound [14], the schedulability of the above example can be guaranteed when $\frac{C_1^1}{T_1^1}+\frac{C_2^1}{T_2^1}+\frac{C_2^2}{T_2^2} \leq 3(2^{\frac{1}{3}}-1) = 0.779$ and accordingly $\epsilon \leq \frac{0.779}{3}$. Nevertheless, the schedulability bound is inversely proportional to the total number of modes in a system, and it thus cannot be put into practice.

To the best of our knowledge, there is no previous work studying on the mode change system under FPM.

A. Carry-in Effect under FPM Scheduling

Unfortunately, under FPM scheduling, the critical instant provided by Lemma 1 is no longer satisfied, and the so-called *carry-in* effect has to thus be considered.

Example. Consider a system with two tasks: a sporadic task $\tau_1 = \{(10, 30, 30)\}$ and a multi-mode task $\tau_2 = \{(5, 10, 10), (16, 30, 30)\}$. The priority assignment accords with RM scheduling. As shown in Figure 2, task τ_1^2 misses its deadline at time 40, whereas it will meet its deadline if released as the *synchronous arrival sequence* mentioned in Lemma 1.

B. Sufficient Test for FPM Scheduling

In this section, we first use the concept of problem window extension to quantify the carry-in job and then reuse the technique derived in the previous section for FPM scheduling.

Consider any legal sequence of jobs of task system τ, on which a deadline miss occurs. Without loss of generality, we assume mode τ_k^h is the mode that there exist $k-1$ tasks such that each task has at least one mode that is assigned higher priority than mode τ_k^h. Suppose that mode τ_k^h is the one to first miss a deadline and arrives at time-instant t_a, and this deadline miss occurs at time-instant $t_a + D_k^h$.

We now discard all those jobs that have priority lower than τ_k^h's priority from this sequence. Since those jobs with priority lower than τ_k^h's have no effect on the scheduling of the jobs with priority higher than or equals to task τ_k^h, this schedule will see a deadline miss of mode τ_k^h at time-instant $t_a + D_k^h$, and it will also be the first deadline miss in the schedule. Hence in this section, we consider only such a *reduced* sequence of jobs.

Let t_0 denote the latest time-instant with $t_0 \leq t_a$ at which the processor is idle (or executing some lower-priority jobs before discarding), c.f. Figure 3. Clearly, t_0 exists and is well-defined. Let $A_k^h = t_a - t_0$ and $t_d = t_a + D_k^h$. The technique for extending the interval of interest is needed for identifying the necessary condition for a deadline miss where the carry-in effect may occur. We have the following observation:

- Let $\mathcal{X}$ denote the workload contributed from task τ_k, that is to under anlaysis, over $[t_0, t_d)$. Due to mode τ_k^h's deadline miss, the amount of execution time, executed for mode τ_k^h, is strictly less than C_k^h over $[t_a, t_d)$. By the definition of t_0, the jobs of task τ_k that contribute to the interval $[t_0, t_d)$ must arrive no earlier than t_0. The jobs of task τ_k contributing to $[t_0, t_a)$ is bounded by the multi-mode demand bound function of task τ_i. Hence, we have
$$\mathcal{X} \leq C_k^h + DBF_i(t_a - t_0, \tau_k^h) \tag{7}$$
- The higher-priority jobs of any multi-mode task must arrive no earlier than time-instant t_0. Hence, the work executing prior to the last release time t_i is bounded from above by its multi-mode demand bound function.

Lemma 6. *If task mode τ_k^h misses its deadline under FPM upon an assignment π, it must be the case that there exist an $A_k^h \geq 0$ and an assignment $t_1, t_2, \ldots, t_{k-1}$ of the last release of task τ_i such that $\forall i \in \{1, ..., k-1, k\}$,*

$$\mathcal{X} + \sum_{\tau_j \in hp(\tau_k^h)} DBF_j(t_j - t_0, \tau_k^h) + \sum_{j:t_j \leq t_i} C_j^{max}(\tau_k^h) > t_i - t_0$$

and for time t_d

$$\mathcal{X} + \sum_{\tau_j \in hp(\tau_k^h)} DBF_j(t_j - t_0, \tau_k^h) + \sum_{j=1}^{k-1} C_j^{max}(\tau_k^h) > A_k^h + D_k^h$$

Proof: This is by the above discussions with a similar proof as in Lemma 3. ∎

Without loss of generality, we can simply set t_0 to 0.

Theorem 2 (QT-FPM)**.** *Task mode* τ_k^h *is schedulable under any FPM scheduling if*

$$D_k^h - \sum_{i=1}^{k-1} C_i^{max}(\tau_k^h) - C_k^h \geq 0 \tag{8}$$

and

$$C_k^h \geq D_k^h - \sum_{i=1}^{k-1} \left(U_i^{max}(\tau_k^h) \cdot t_i^*\right) - \sum_{i=1}^{k-1} C_i^{max}(\tau_k^h) \tag{9}$$

where

$$t_i^* = D_k^h - \sum_{j=i}^{k-1} C_j^{max}(\tau_k^h)$$

in which higher-priority tasks τ_i *are indexed by non-increasing* $\beta_i(\tau_k^h)$.

Proof: In case of Eq. (8), it is clear that there will be a deadline miss.

By Lemma 6 and Eq. (7) and using the same concept of Lemma 4 and Lemma 5, we then have the necessary condition for the deadline miss under FPM:

$$C_k^h > D_k^h - \sum_{i=1}^{k-1} \left(U_i^{max}(\tau_k^h) \cdot t_i^*\right) - \sum_{i=1}^{k-1} C_i^{max}(\tau_k^h) + A_k^h \left(1 - \sum_{i=1}^{k} U_i^{max}(\tau_k^h)\right) \tag{10}$$

where

$$t_i^* = D_k^h - \sum_{j=i}^{k-1} C_j^{max}(\tau_k^h)$$

Notice that due to the assumption that $U^{sum} \leq 1$, the RHS of the above inequality is monotonically increasing with the length of interval A_k^h. By substituting $A_k^h = 0$ and taking the negation of Eq. (10), this theorem is proved. ∎

Roughly speaking, the work contributing over interval $[t_0, t_d)$ where $t_0 \neq t_a$ will be no more than that in the case $t_0 = t_a$. One may observe that in order to make the interval $[t_0, t_a)$ busy, there will be a loss of some workloads over $[t_a, t_d)$ due to $U^{sum} \leq 1$ according to our analysis.

C. Utilization Bounds for Implicit-Deadline Tasks under RM Scheduling

In this section we specifically study RM scheduling in implicit-deadline multi-mode systems where the relative deadline of each mode is equal to its period. Under RM scheduling, we first observe that the interference from the last release can be naturally bounded:

Lemma 7. *Under RM scheduling, for any task* τ_i *with respect to mode* τ_k^h

$$C_i^{max}(\tau_k^h) \leq U_i^{max}(\tau_k^h) \cdot T_k^h$$

Proof: Under RM scheduling, only those modes with period $T_a^b \leq T_k^h$ will be assigned higher priority than τ_k^h. By the definition of $C_i^{max}(\tau_k^h)$ we have that

$$C_i^{max}(\tau_k^h) = \max_{\tau_a^b \in hp(\tau_k^h) \cap \tau_i} \{U_a^b \cdot T_a^b\} \leq \quad U_i^{max}(\tau_k^h) \cdot T_k^h$$

∎

By Lemma 7 we now pessimistically inflate each $C_i^{max}(\tau_k^h)$ to $U_i^{max}(\tau_k^h) \cdot T_k^h$. Note that thereafter we can simply relax the ordering specified in Theorem 2 since according to Theorem 2 for all tasks τ_i

$$\beta_1 = \beta_2 = ... = \beta_{k-1} = \frac{C_i^{max}(\tau_k^h)}{U_i^{max}(\tau_k^h)} = T_k^h$$

after the inflation. Theorem 3 immediately follows:

Theorem 3. *Implicit-deadline mode* τ_k^h *is schedulable under RM if*

$$U_k^h \leq 1 - 2\sum_{i=1}^{k-1} U_i^{max}(\tau_k^h) + \frac{1}{2}\left(\sum_{i=1}^{k-1} U_i^{max}(\tau_k^h)\right)^2 + \frac{1}{2}\sum_{i=1}^{k-1}\left(U_i^{max}(\tau_k^h)\right)^2 \tag{11}$$

Proof: In case of Eq. (8), due to $U^{sum} \leq 1$, we know that $T_k^h \cdot \left(1 - \sum_{i=1}^{k-1} U_i^{max}(\tau_k^h) - U_k^h\right) \geq 0$. For notational simplicity, we denote U_i as $U_i^{max}(\tau_k^h)$ in this proof. In case of Eq. (9), substituting C_i^{max} in Eq. (9) by $U_i^{max} \cdot T_k^h$ and D_k^h by T_k^h, we have

$$\begin{aligned}
\frac{C_k^h}{T_k^h} &\leq 1 - \sum_{i=1}^{k-1}\left(U_i \cdot \left(1 - \sum_{j=i}^{k-1} U_j\right)\right) - \sum_{i=1}^{k-1} U_i \\
&= 1 - 2\sum_{i=1}^{k-1} U_j + \sum_{i=1}^{k-1}\left(U_i \cdot \left(\sum_{j=i}^{k-1} U_j\right)\right) \\
&= 1 - 2\sum_{i=1}^{k-1} U_i + \frac{1}{2}\left(\sum_{i=1}^{k-1} U_i\right)^2 + \frac{1}{2}\sum_{i=1}^{k-1} U_i^2
\end{aligned}$$

Hence this theorem is proved. ∎

The schedulability test proposed in the previous section must check each task mode τ_k^h individually. If one is willing to sacrifice some precision, we can further provide two linear-time schedulability tests for a task set, called *quadratic bound* (QB) and *total utilization bound*, in Theorem 4 and 5, respectively, in which the proofs are in Appendix.

Theorem 4 (QB-RM)**.** *A multi-mode task set* τ *with implicit-*

deadline is schedulable under RM scheduling if

$$U_a \leq 1 - 2 \sum_{\tau_i \in \tau \setminus \{\tau_a\}} U_i^{max} + \frac{1}{2} \left(\sum_{\tau_i \in \tau \setminus \{\tau_a\}} U_i^{max} \right)^2 + \frac{1}{2} \sum_{\tau_i \in \tau \setminus \{\tau_a\}} (U_i^{max})^2 \quad (12)$$

where $U_a = min_{\tau_i \in \tau}\{U_i^{max}\}$.

Proof: It is clear that for any task $\tau_k \in \tau$ if

$$U_k^{max} \leq 1 - 2 \sum_{\tau_i \in \tau \setminus \{\tau_k\}} U_i^{max} + \frac{1}{2} \left(\sum_{\tau_i \in \tau \setminus \{\tau_k\}} U_i^{max} \right)^2 + \frac{1}{2} \sum_{\tau_i \in \tau \setminus \{\tau_k\}} (U_i^{max})^2 \quad (13)$$

then Eq. (11) must hold for all modes $\tau_k^h \in \tau_k$.

We then show that if Eq. (13) holds by the choice of $U_k^{max} = min_{\tau_i \in \tau}\{U_i^{max}\}$, then it also holds for all the other cases.

Let τ_a denote the task with minimum U_i^{max} among all tasks. Given that Eq. (13) holds for τ_a, we have that

$$U_a^{max} \leq 1 - 2 \sum_{\tau_i \in \tau \setminus \{\tau_a\}} U_i^{max} + \frac{1}{2} \left(\sum_{\tau_i \in \tau \setminus \{\tau_a\}} U_i^{max} \right)^2 + \frac{1}{2} \sum_{\tau_i \in \tau \setminus \{\tau_a\}} (U_i^{max})^2 \quad (14)$$

We prove this by contradiction: *suppose for some* $U_b^{max} \geq U_a^{max}$ *Eq.* (12) *does not hold*:

$$U_b^{max} > 1 - 2 \sum_{\tau_i \in \tau \setminus \{\tau_b\}} U_i^{max} + \frac{1}{2} \left(\sum_{\tau_i \in \tau \setminus \{\tau_b\}} U_i^{max} \right)^2 + \frac{1}{2} \sum_{\tau_i \in \tau \setminus \{\tau_b\}} (U_i^{max})^2 \quad (15)$$

For notational simplicity, let U_a^{max} and U_b^{max} denote U_a and U_b. Summing the above two inequities we have

$$\begin{aligned} U_a - U_b &\leq 2(U_a - U_b) + \frac{1}{2}\left(U_a^2 - U_b^2\right) \\ &\quad + \frac{1}{2}\left(U_a + U_b + 2 \sum_{\tau_i \in \tau \setminus \{\tau_a, \tau_b\}} U_i^{max}\right)(U_b - U_a) \\ &= (U_a - U_b)\left(1 - \sum_{\tau_i \in \tau \setminus \{\tau_a, \tau_b\}} U_i^{max}\right) \\ \equiv \sum_{\tau_i \in \tau \setminus \{\tau_a, \tau_b\}} U_i^{max} &\geq 1 \end{aligned}$$

which contradicts the fact that $U^{sum} \leq 1$ and $U_a, U_b > 0$. Hence this theorem is proved. ∎

Theorem 5. *An implicit-deadline multi-mode tasks system* τ *is schedulable under RM scheduling if*

$$U^{sum} \leq \begin{cases} \dfrac{2(n-1) - \sqrt{2(n-1)(n-2)}}{n} & \text{if } n \geq 3 \quad (16) \\ \dfrac{1}{2} + \dfrac{1}{2n} & \text{if } n = 2 \quad (17) \end{cases}$$

Proof: Our objective in this proof is to find the minimum U^{sum} such that the equality of Eq. (12) always holds. For $n = 2$ the minimization can be done directly by solving the differential equation in two variables. We discuss the case for $n \geq 3$ by using Lagrange Multiplier Method.

$$\text{minimize} \quad \sum_{i=1}^{n} U_i$$

$$\text{subject to} \quad U_n = 1 - 2\sum_{i=1}^{n-1} U_i + \frac{1}{2}(\sum_{i=1}^{n-1} U_i)^2 + \frac{1}{2}\sum_{i=1}^{n-1} U_i^2 \quad (18a)$$

$$U_n \geq 0 \quad (18b)$$

Let λ be the multiplier of the schedulability constraint by Eq. (18a) and u_n be the multiplier of the constraint $U_n \geq 0$. The Lagrange function is

$$L(U_1, U_2, ..., U_n) = \sum_{i=1}^{n} U_i + u_n(-U_n) + \lambda\left(1 - 2\sum_{i=1}^{n-1} U_i + \frac{1}{2}(\sum_{i=1}^{n-1} U_i)^2 + \frac{1}{2}\sum_{i=1}^{n-1} U_i^2 - U_n\right)$$

with derivatives

$$\frac{\partial L}{\partial U_i} = \begin{cases} 1 - u_n - \lambda, & \text{if } i = n \quad (19a) \\ 1 + \lambda\left(-2 + U_i + \sum_{i=1}^{n-1} U_i\right) & \text{otherwise} \quad (19b) \end{cases}$$

A necessary condition for the minimum is that the two derivatives of (19a) and (19b) are zero. To solve these equations, we look at several cases:

Case 1: $u_n = 0$

When $u_n = 0$, $\lambda = 1$ by Eq. (19a) according to the necessary condition for the minimum. In addition, Eq. (19b) implies that for all $i \neq n$

$$U_1 = U_2 = ... = U_{n-1} \quad (20)$$

Setting $\lambda = 1$ in Eq. (19b) and using the above condition, we obtain optimum values U_i $\forall 1 \leq i \leq n-1$:

$$U_i = \frac{n-1}{n} \quad (21)$$

which leads to the violation of the nonnegative constraint $U_n \geq 0$ after solving the condition of (18a).

Case 2: $u_n \neq 0$

When $u_n \neq 0$, it must be the case $U_n = 0$ due to the complementarity.

After solving Eq. (18a) with the condition (20), we get for all $i \neq n$

$$U_i = \frac{2(n-1) - \sqrt{2(n-1)(n-2)}}{n} \cdot \frac{1}{n-1} \quad (22)$$

and

$$U_n = 0$$

It follows that

$$\sum_{i=1}^{n} U_i = \frac{2(n-1) - \sqrt{2(n-1)(n-2)}}{n} \quad (23)$$

■

Theorem 6. *An implicit-deadline multi-mode tasks system τ is schedulable under RM scheduling if*

$$U^{sum} \leq 2 - \sqrt{2} \approx 0.5857$$

Proof: By taking $n \to \infty$ for $n \geq 3$ in Theorem 5, Theorem 6 follows immediately. ■

Comparing the total utilization bounds and quadratic bounds: Figure 4 illustrates the difference between QB and the total utilization bound in Theorem 4 and in Theorem 5, respectively, for two tasks. We can clearly see that the feasibility region below QB is larger than that below the total utilization bound, in Figure 4a. Moreover, as shown in Figure 4b, the summation of utilizations that can be schedulable under QB is larger, especially when U_1 is away from 0.5.

It is not difficult to see that QB and the total utilization bound are tight in Theorem 4 and in Theorem 5, respectively. The idea is that increasing the maximum utilization of any higher-priority task mode τ_i leads to mode τ_n^h's deadline miss. We illustrate this with the following example.

Example. For a given n, there are two modes in τ_i for $i = 1, 2, \ldots, n-1$ with given $U_i^{max} = \frac{2(n-1)-\sqrt{2(n-1)(n-2)}}{n}$. $\frac{1}{n-1}$, in which T_i^2 is T_n^h, C_i^2 is $T_n^h \cdot U_i^{max}$, T_i^1 is $T_n^h \cdot \left(1 - \sum_{j=i}^{n-1} U_j^{max}\right)$, and C_i^1 is $T_i^1 \cdot U_i^{max}$. Thus, task mode τ_n^h can only be schedulable when its utilization is 0.

D. Computational complexity.

Considering all deadlines to be met, QT-RM runs in $O(n \log n \cdot (M_1 + M_2 + ... + M_n) + (M_1 + M_2 + ... + M_n) log(M_1 + M_2 + ... + M_n))$ where $n \log n$ comes from the sorting runtime for the load factor and $(M_1 + M_2 + ... + M_n) log(M_1 + M_2 + ... + M_n)$ is the sorting time for prioritizing the modes at the beginning. On the other hand, Theorem 3 can be computed in $O(n(M_1 + M_2 + ... + M_n) + (M_1 + M_2 + ... + M_n) log(M_1 + M_2 + ... + M_n))$ whereas QB-RM requires only $O(n + (M_1 + M_2 + ... + M_n))$ where $(M_1 + M_2 + ... + M_n)$ is accounted for deciding the maximum utilization among modes.

VI. Evaluations

As can be seen, we have established several utilization-based tests for FPT and RM scheduling. In this section we evaluate the effectiveness of the existing tests and the proposed utilization-based tests in terms of the number of tasksets that are deemed schedulable. First, we recap these tests as follows:

- Demand-based Test under FPT (DT-FPT): the time-demand test approach under FPT presented in Section III.D in [10].
- Quadratic Test under FPT (QT-FPT): Theorem 1.
- Quadratic Bound under RM (QB-RM): Theorem 4.
- Quadratic Test under RM (QT-RM): RM is an example of QT-FPM test of Theorem 2.

The metric to compare the results is to measure the *success ratio* of these tests for a given goal of task set utilization. We generate 100 task sets for each utilization level. The success ratio of a level is said to be the number of task sets that are schedulable divided by the number of task sets for this level, i.e. 100.

A. Task Set Generation

The task set was generated in a similar manner to the method in [10], for testing the variable rate-behavior (VRB) task models, as mentioned in Section I. We first generated a set of sporadic tasks. The cardinality of the task set was 10. The UUniFast method [6] was adopted to generate a set of utilization values with the given goal. We here use the approach presented by Davis and Burns [11] to generate task periods according to an exponential distribution. Here the two orders of magnitude to control the period values between largest and smallest periods are explored. (i.e., $1 - 10ms$, $10 - 100ms$). Task relative deadlines were implicit. The worst-case execution time was computed accordingly, i.e. $C_i = T_i U_i$.

We converted a proportion p of sporadic tasks to multi-mode tasks as follow:

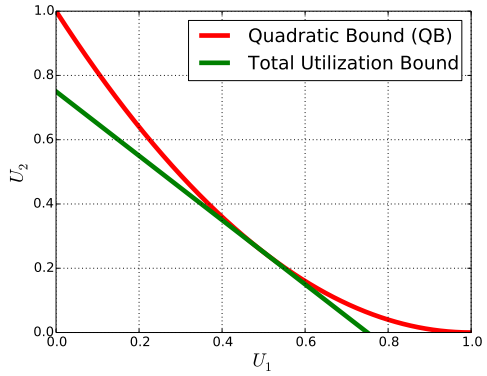
- A multi-mode task has M execution modes
- The generated sporadic task triplet (C_i, T_i, D_i) was assigned to the setting of task mode τ_i^1.
- We use a scaling factor 1.5 to assign the parameters of the other modes[3], i.e., $C_i^{m+1} = 1.5 C_i^m$, $T_i^{m+1} = 1.5 T_i^m$.
- We randomly choose a mode to have the largest utilization. The worst-case execution times of the remaining modes were adjusted by multiplying them by uniform random values in the range $[0.75, 1]$.

Checking the FPT feasibility of a multi-mode task set was achieved by using the method for sporadic tasks, called *Audsley's Algorithm* [3], since the above tests comply with the required conditions for the compatibility provided in [9]. Due to the page limitation, we report only the results evaluating a variety of M and p in the following subsection.

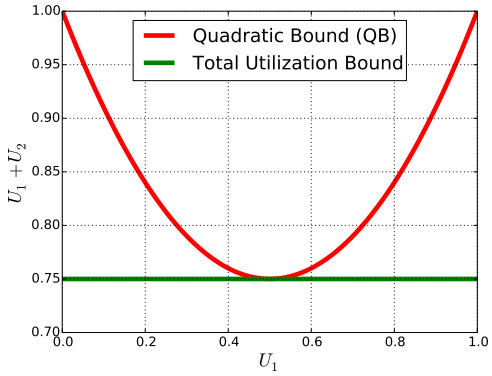
B. Experimental Results

We first evaluated the impact on different numbers of modes when p was set to 0.5, as illustrated in Figure 5a. With a less number of modes, the schedulability of multi-mode tasks under FPT scheduling can be provided with up to 90% of utilizations; however, it drops significantly when the number of modes increases. This also accords to the discussion in previous section: the FPT scheduling may perform rather poorly. In fact, under FPT, some jobs with sizable WCET released by different tasks may dominate each other, and this thus results in poor performance of the schedulability. On the other hand,

[3]We use the dynamic programming approach to implement DT-FPT instead of using the ILP solver for the sake of efficiency. To comply with it, we assume the discrete time model for periods. We here apply a correction factor $s_i^m = \frac{\lceil T_i \rceil}{T_i}$ on both the generated period and execution time to assure this assumption.

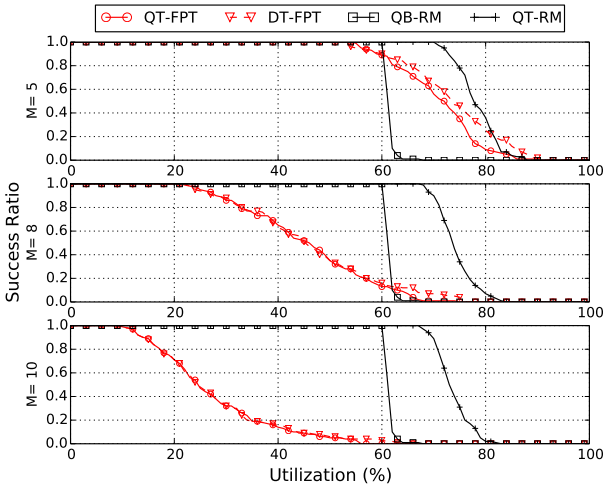


(a) Utilization of U_2 under given U_1

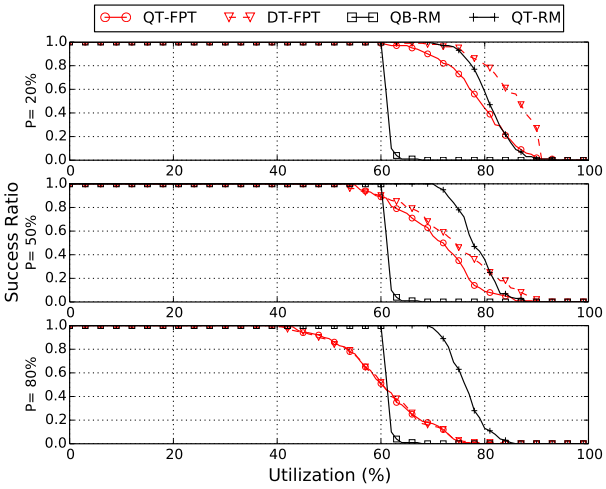


(b) Total utilization $U_1 + U_2$ under given U_1

Figure 4: Utilization Bounds for 2 implicit-deadline multi-mode tasks under RM



(a) Success ratio with different M when $p = 50\%$.



(b) Success ratio with different p when $M = 5$.

Figure 5: Effectiveness evaluation in terms of the number of tasksets that are deemed schedulable.

RM scheduling is expected to be more sustainable due to the essential utilization bound guarantee. Moreover, QT-RM can accept the task set with total utilizations of up to 80% compared to only 65% by QB-RM. However, QB-RM takes the advantage on the runtime overhead and can be applied on-line efficiently when long execution times are prohibitive.

For FPT scheduling, the proposed QT-FPT is comparable to the pseudo-polynomial-time DT-FPT. With a number of modes, the workload we overly approximate before the last release may be very close to the actual workload. One can imagine that a task with a larger number of modes is more flexible to generate a worse release sequence for interfering the lower priority one.

We further evaluate the effectiveness with different proportions of multi-mode tasks when $M = 5$. Similar results can be observed in Figure 5b.

VII. Conclusions

This paper addresses the scheduling problem of mode change real-time tasks under fixed-priority scheduling. Simulation results show that our proposed tests are comparable with the pseudo-polynomial-time demand-based test under FPT. To the best of our knowledge, this is the first work providing the utilization-based test for RM scheduling for multi-mode tasks. Empirical results show that our proposed tests for RM scheduling can accept test sets with utilizations of up to 80%.

Although we focus ourselves on the schedulability tests on multi-mode tasks, the utilization-based tests can also be used to analyze GMF tasks [4], digraph model [24], since the multi-mode task model has complete freedom to change modes, whereas the GMF task model and digraph model have some additional constraints. Our proposed tests in Theorems 5 and 6 for RM scheduling can also be adopted on aperiodic models, as proposed in [1], by relaxing the finite number of modes and are able to accept more tasks during on-line changes.

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