technische universität
dortmund

fakultät für informatik
informatik 12

Petri Nets

Peter Marwedel
TU Dortmund,
Informatik 12

2008/10/10

Generalization of data flow: Computational graphs Example: Petri nets

Introduced in 1962 by Carl Adam Petri in his PhD thesis.

Focus on modeling causal dependencies;

no global synchronization assumed (message passing only).

Key elements:

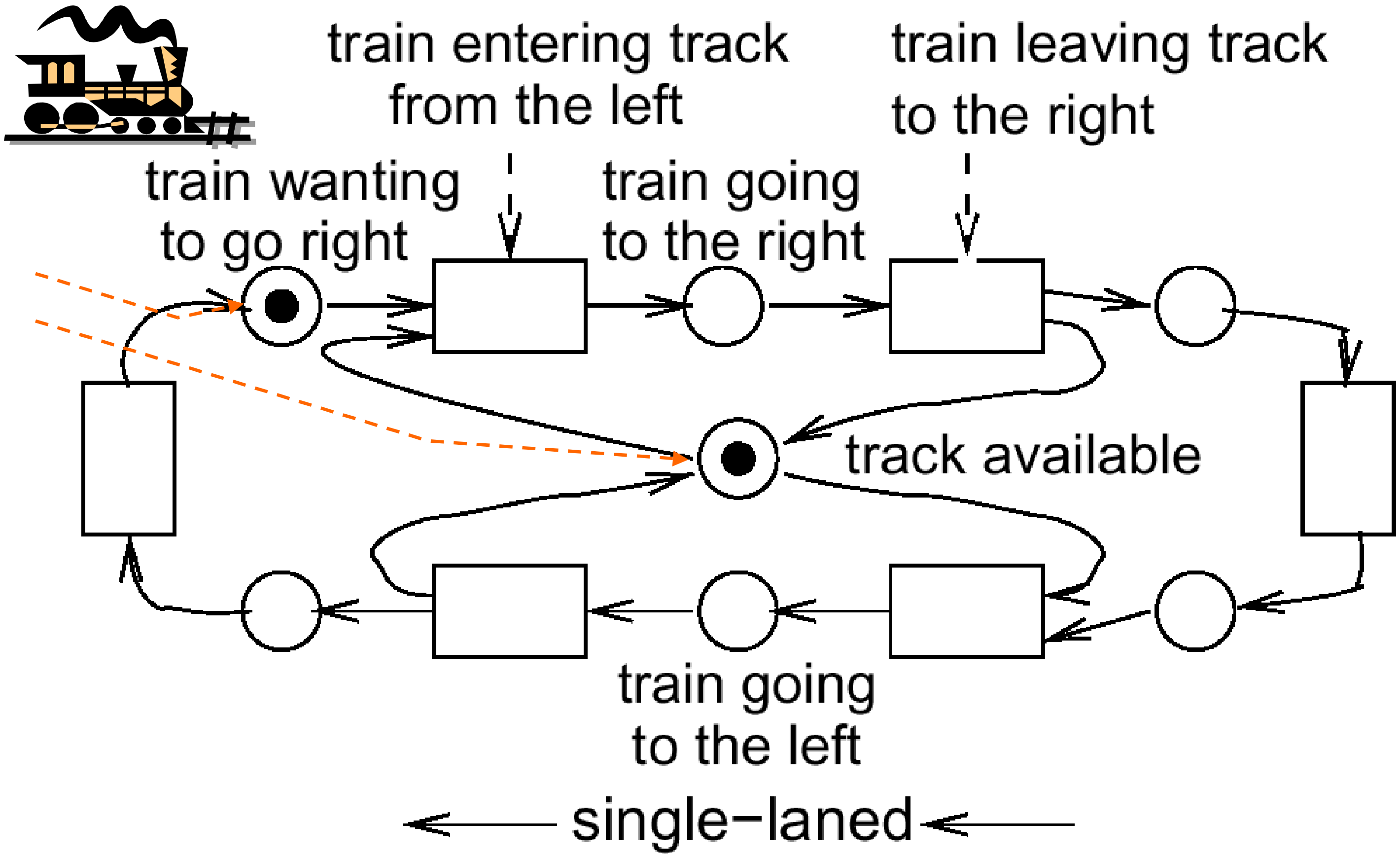
- **Conditions**
  Either met or no met.
- **Events**
  May take place if certain conditions are met.
- **Flow relation**
  Relates conditions and events.

Conditions, events and the flow relation form

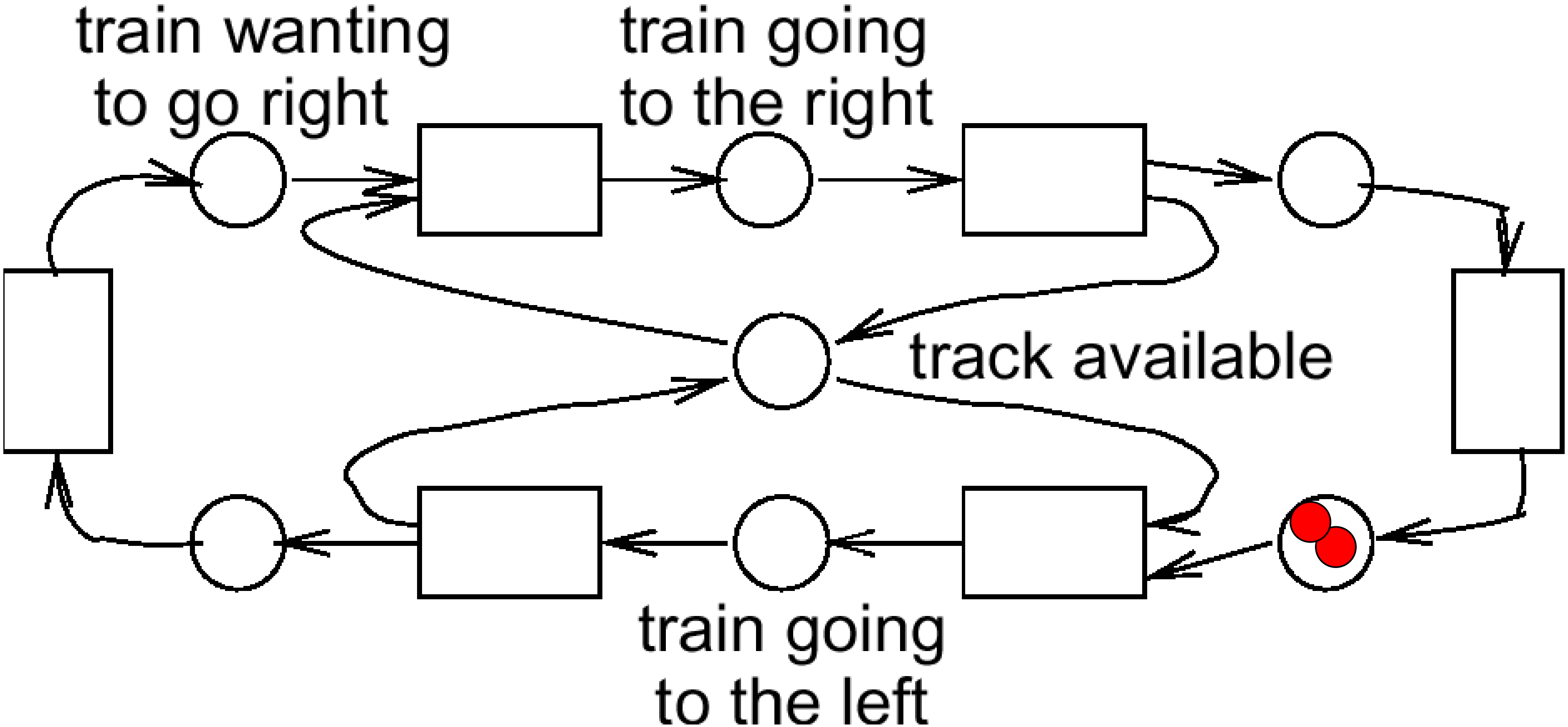
a **bipartite graph** (graph with two kinds of nodes).

Example: Synchronization at single track rail segment

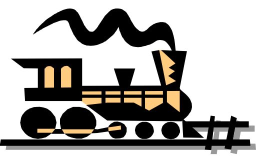
„Preconditions“

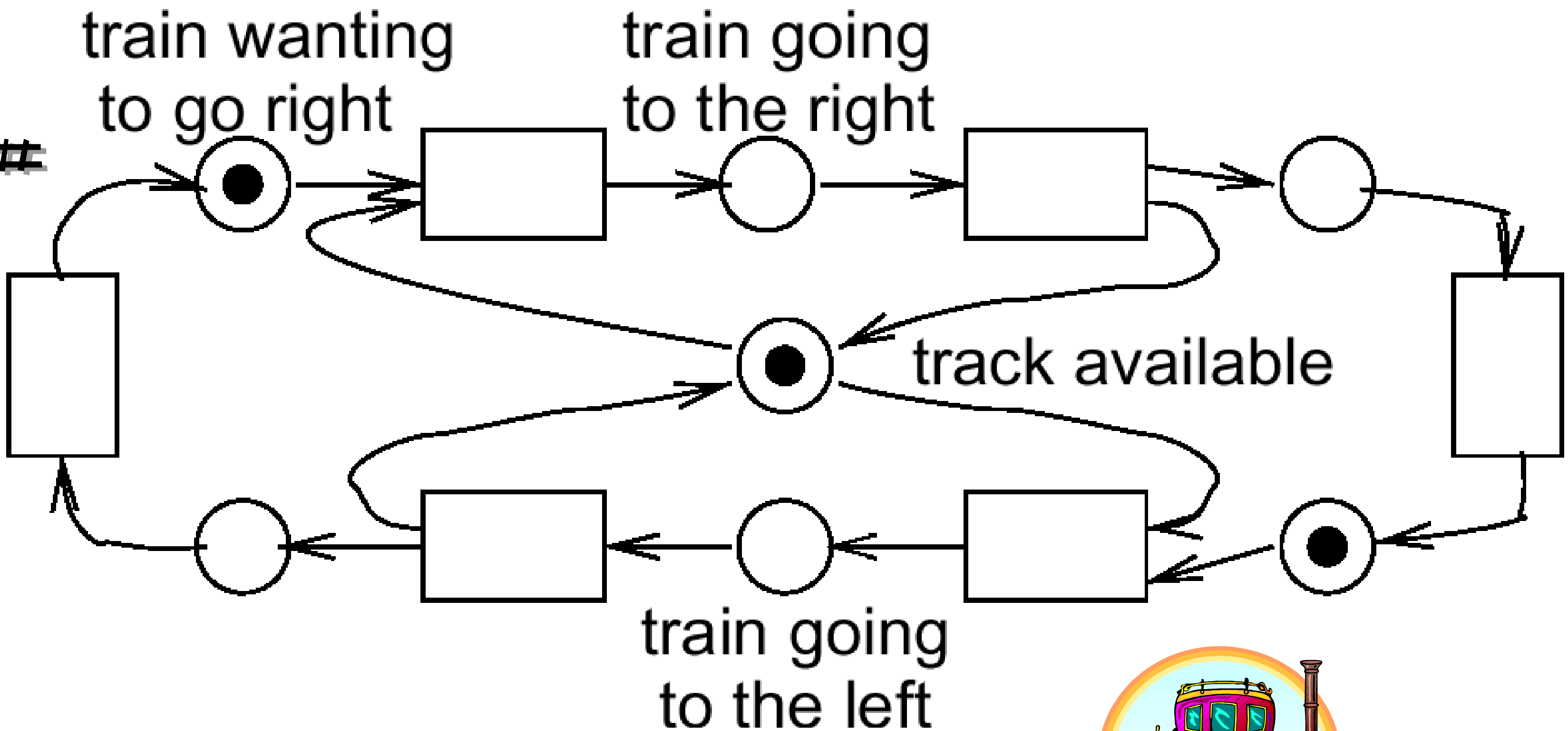


Playing the „token game“



Conflict for resource „track“





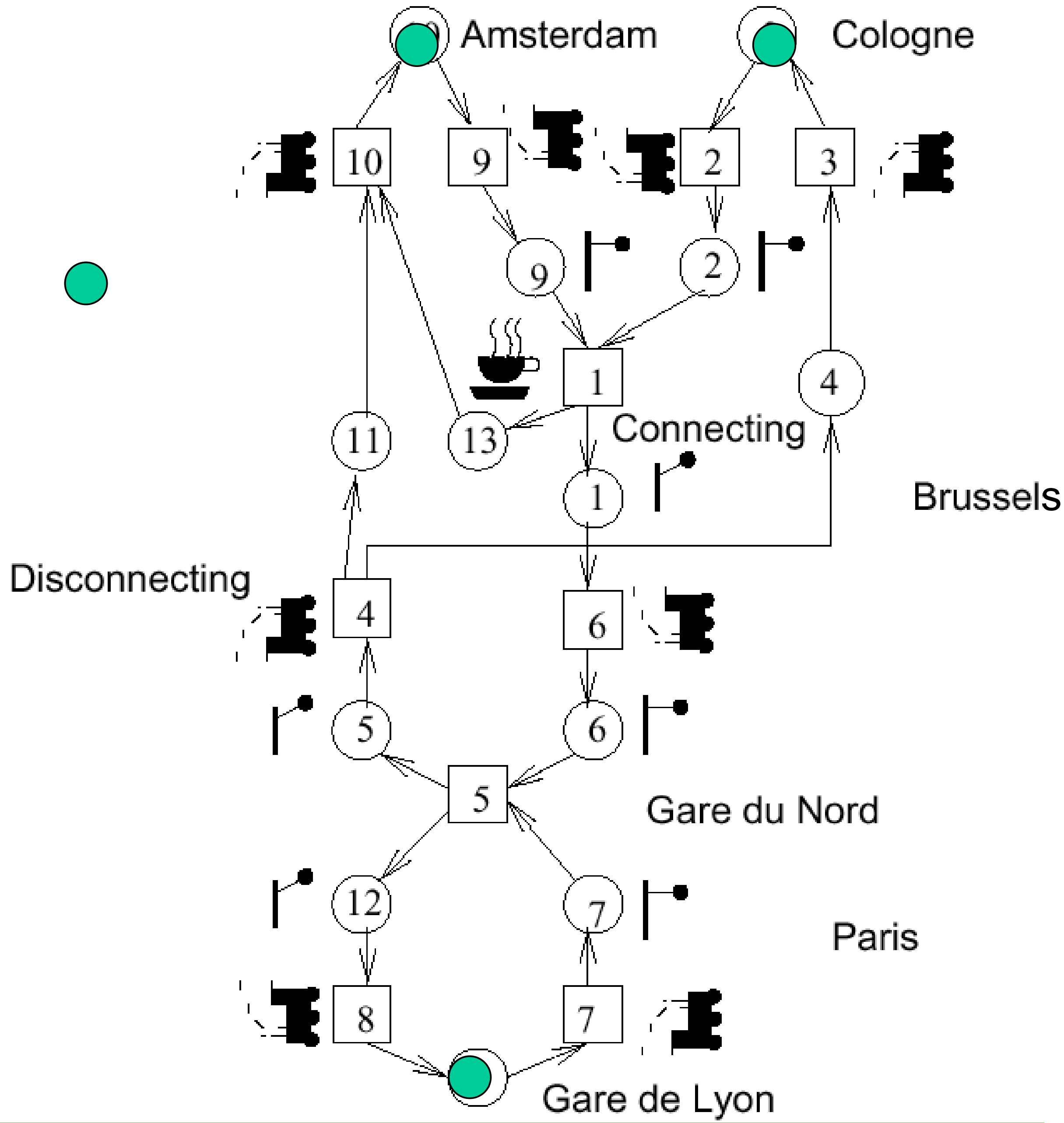
More complex example (1)

Thalys trains between Cologne, Amsterdam, Brussels and Paris.

[http://www.thalys.com/be/en]

More complex example (2)

Slightly simplified: Synchronization at Brussels and Paris, using stations “Gare du Nord” and “Gare de Lyon” at Paris



Condition/event nets

Def.: N=(C,E,F) is called a **net**, iff the following holds

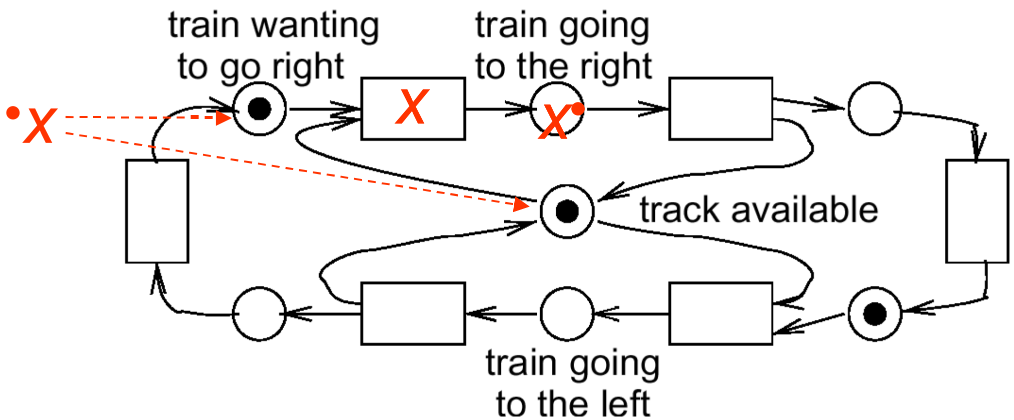
2. C and E are disjoint sets
3. $F \subseteq (C \times E) \cup (E \times C)$; is binary relation, („**flow relation**“)

Def.: Let N be a net and let $x \in (C \cup E)$.

$^\bullet x := \{y \mid y\ F\ x\}$ is called the set of **preconditions.**

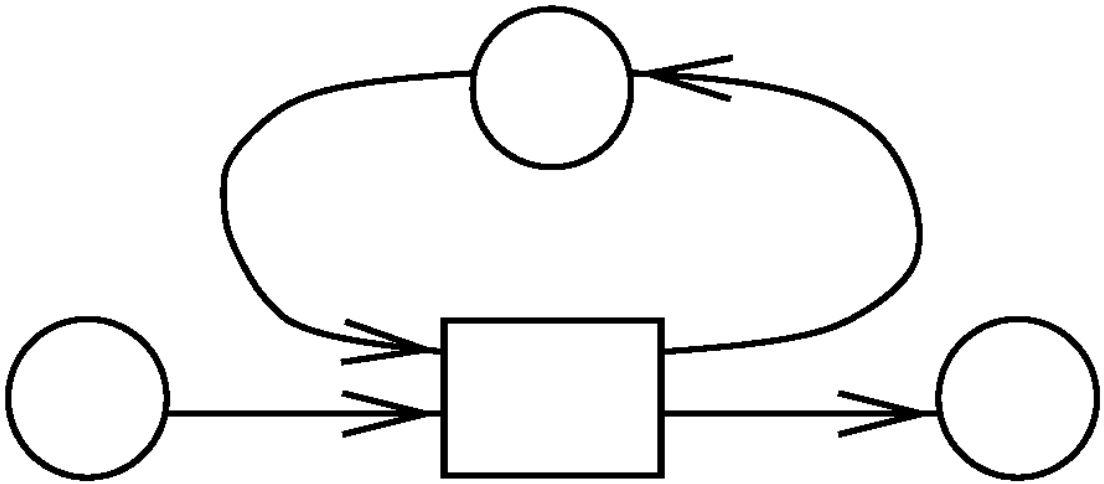
$x^\bullet := \{y \mid x\ F\ y\}$ is called the set of **postconditions.**

Example:



Loops and pure nets

Def.: Let $(c,e) \in C \times E$. (c,e) is called a **loop** iff $cFe \wedge eFc$.

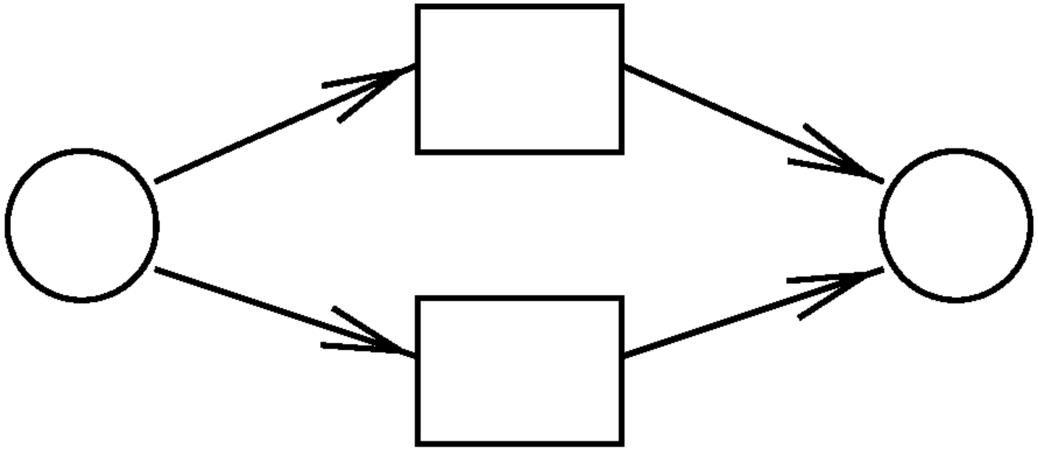


Def.: Net $N=(C,E,F)$ is called **pure**, if F does not contain any loops.

Simple nets

Def.: A net is called **simple** if no two transitions *t1* and *t2* have the same sets of input and output places.

Example (not a simple net):

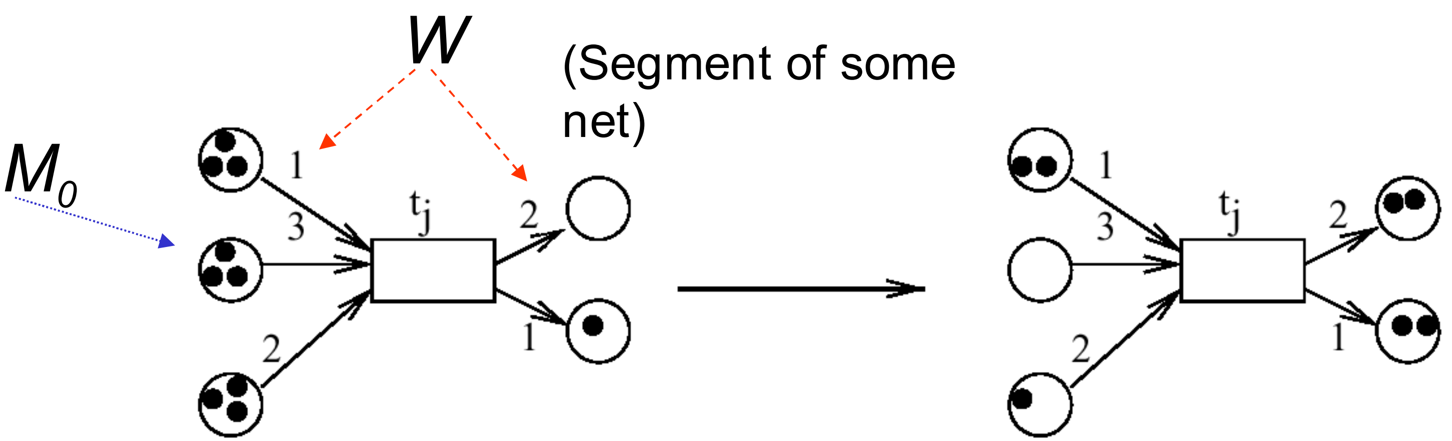


Def.: Simple nets with no isolated elements meeting some additional restrictions are called **condition/event nets (C/E nets)**.

Place/transition nets

Def.: (P, T, F, K, W, M_0) is called a **place/transition net** iff

2. $N=(P,T,F)$ is a **net** with places $p \in P$ and transitions $t \in T$
3. $K: P \rightarrow (\mathbb{N}_0 \cup \{\omega\}) \setminus \{0\}$ denotes the **capacity** of places
   (ω symbolizes infinite capacity)
4. $W: F \rightarrow (\mathbb{N}_0 \setminus \{0\})$ denotes the **weight of graph edges**
5. $M_0: P \rightarrow \mathbb{N}_0 \cup \{\omega\}$ represents the **initial marking** of places

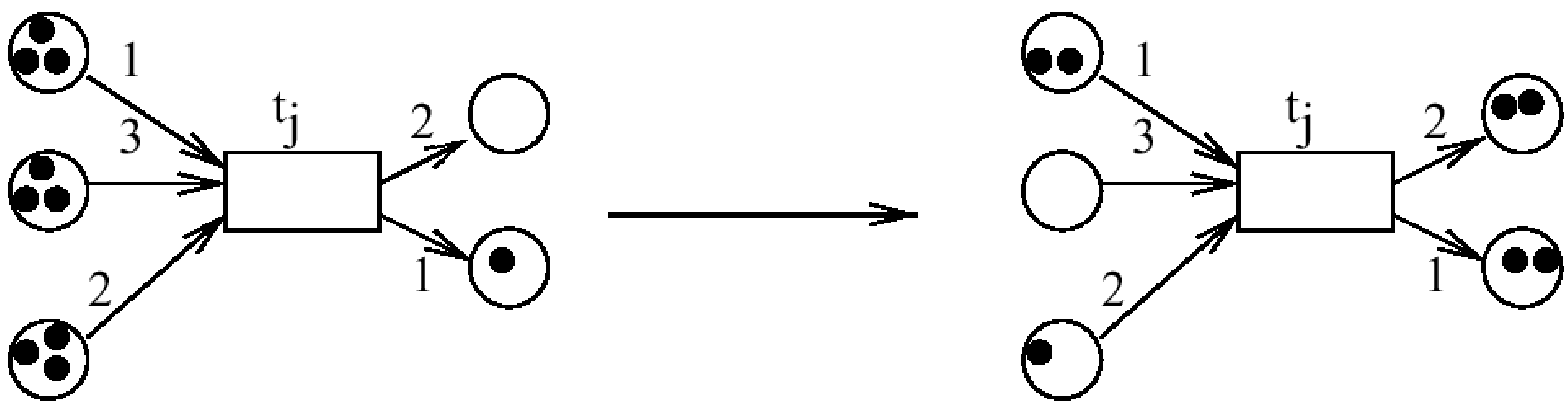


defaults:
$K = \omega$
$W = 1$

Computing changes of markings

„Firing“ transitions *t* generate new markings on each of the places *p* according to the following rules:

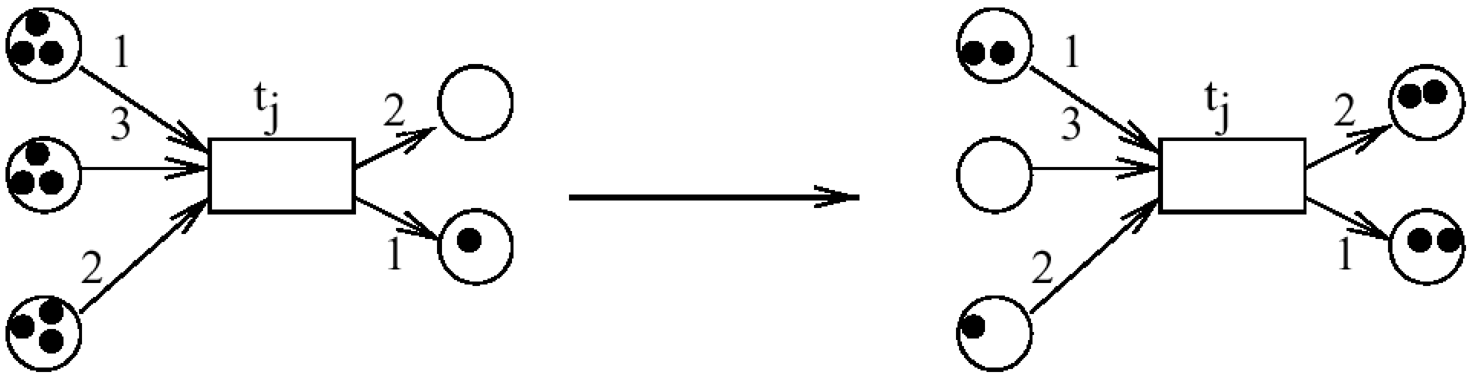
$$M'(p) = \begin{cases} M(p) - W(p,t), & \text{if } p \in {}^{\bullet}t \setminus t^{\bullet} \\ M(p) + W(t,p), & \text{if } p \in t^{\bullet} \setminus {}^{\bullet}t \\ M(p) - W(p,t) + W(t,p), & \text{if } p \in {}^{\bullet}t \cap t^{\bullet} \\ M(p) & \text{otherwise} \end{cases}$$



Activated transitions

Transition *t* is „activated“ iff

$$(\forall p \in {}^{\bullet}t : M(p) \geq W(p,t)) \wedge (\forall p \in t^{\bullet} : M(p) + W(t,p) \leq K(p))$$



Activated transitions can „take place“ or „fire“, but don‘t have to.

We never talk about „time“ in the context of Petri nets.

The order in which activated transitions fire, is not fixed (it is non-deterministic).

Shorthand for changes of markings

Slide 12:
$$M'(p) = \begin{cases} M(p) - W(p,t), & \text{if } p \in {}^\bullet t \setminus t^\bullet \\ M(p) + W(t,p), & \text{if } p \in t^\bullet \setminus {}^\bullet t \\ M(p) - W(p,t) + W(t,p), & \text{if } p \in {}^\bullet t \cap t^\bullet \\ M(p) & \text{otherwise} \end{cases}$$

Let
$$\underline{t}(p) = \begin{cases} -W(p,t) \text{ if } p \in {}^\bullet t \setminus t^\bullet \\ +W(t,p) \text{ if } p \in t^\bullet \setminus {}^\bullet t \\ -W(p,t) + W(t,p) \text{ if } p \in t^\bullet \cap {}^\bullet t \\ 0 \end{cases}$$

$\Rightarrow$ $\forall p \in P$: $M'(p) = M(p) + \underline{t}(p)$

$\Rightarrow$ $M' = M + \underline{t}$ +: vector add

Matrix $\underline{N}$ describing all changes of markings

$$\underline{t}(p)=\begin{cases} -W(p,t) \text{ if } p\in {}^\bullet t \setminus t^\bullet \\ +W(t,p) \text{ if } p\in t^\bullet \setminus {}^\bullet t \\ -W(p,t)+W(t,p) \text{ if } p\in t^\bullet \cap {}^\bullet t \\ 0 \end{cases}$$

Def.: Matrix $\underline{N}$ of net N is a mapping

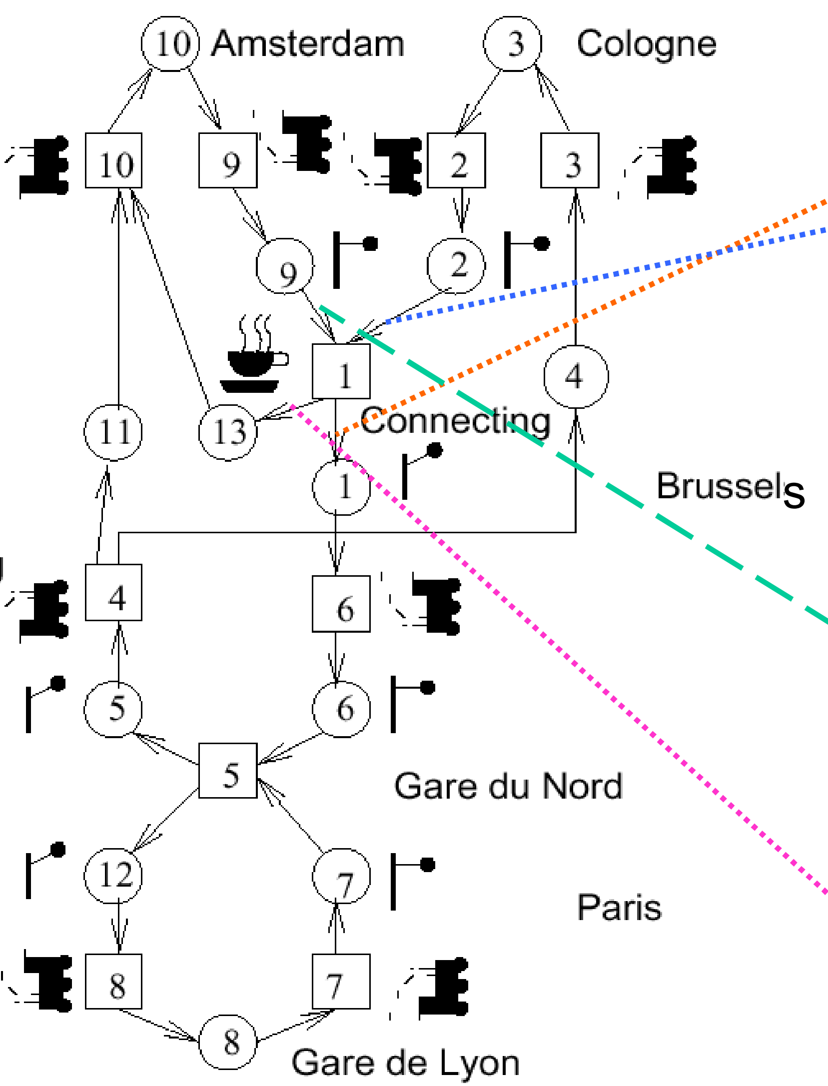
$$\underline{N}: P \times T \rightarrow \mathbb{Z} \text{ (integers)}$$

such that $\forall\, t \in T$: $\underline{N}(p,t)=\underline{t}(p)$

Component in column t and row p indicates the change of the marking of place p if transition t takes place.

For pure nets, ($\underline{N}$, M_0) is a complete representation of a net.

Example: $\underline{N}$ =



	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
p_1	1					-1				
p_2	-1	1								
p_3		-1	1							
p_4			-1	1						
p_5				-1	1					
p_6					-1	1				
p_7					-1		1			
p_8							-1	1		
p_9	-1							1	1	
p_{10}									-1	1
p_{11}				1						-1
p_{12}					1			-1		
p_{13}	1									-1

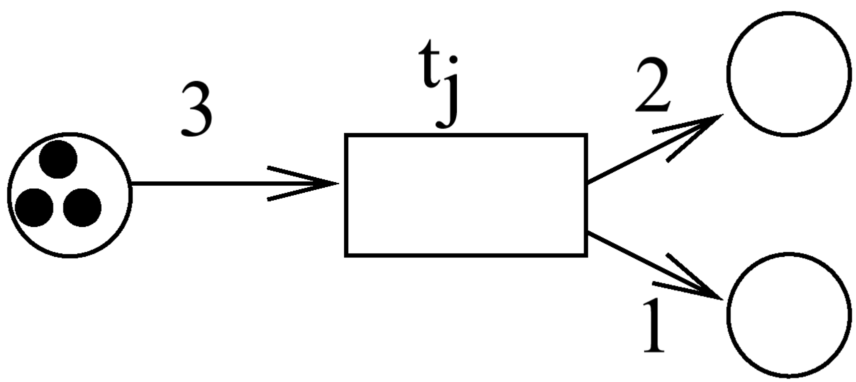
Place - invariants

Standardized technique for proving properties of system models

For any transition $t_j \in T$ we are looking for sets $R \subseteq P$ of places for which the accumulated marking is constant:

$$\sum_{p \in R} \underline{t}_j(p) = 0$$

Example:



Characteristic Vector

$$\sum_{p \in R} \underline{t}_j(p) = 0$$

Let:
$$\underline{c}_R(p) = \begin{cases} 1 \text{ if } p \in R \\ 0 \text{ if } p \notin R \end{cases}$$

$$\Rightarrow 0 = \sum_{p \in R} \underline{t}_j(p) = \sum_{p \in P} \underline{t}_j(p)\,\underline{c}_R(p) = \underline{t}_j \cdot \underline{c}_R$$

Scalar product

Condition for place invariants

$$\sum_{p \in R} \underline{t}_j(p) = \sum_{p \in P} \underline{t}_j(p)\, \underline{c}_R(p) = \underline{t}_j \cdot \underline{c}_R = 0$$

Accumulated marking constant for **all** transitions if

$$\begin{array}{ccc} \underline{t}_1 \cdot \underline{c}_R & = & 0 \\ \ldots & \ldots & \ldots \\ \underline{t}_n \cdot \underline{c}_R & = & 0 \end{array}$$

Equivalent to $\underline{\boldsymbol{N}}^T \boldsymbol{c}_R = \boldsymbol{0}$ where $\underline{\boldsymbol{N}}^T$ is the transposed of $\underline{\boldsymbol{N}}$

More detailed view of computations

$$\begin{pmatrix} \underline{t}_1(p_1)\ldots\underline{t}_1(p_n) \\ \underline{t}_2(p_1)\ldots\underline{t}_2(p_n) \\ \ldots \\ \underline{t}_m(p_1)\ldots\underline{t}_m(p_n) \end{pmatrix} \begin{pmatrix} \underline{c}_R(p_1) \\ \underline{c}_R(p_2) \\ \ldots \\ \underline{c}_R(p_n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

System of linear equations.

Solution vectors must consist of zeros and ones.

Equations with multiple unknowns that must be integers called **diophantic** (☞ Greek mathematician Diophantos, ~300 B.C.).

Diophantic linear equation system more complex to solve than standard system of linear equations (actually NP-hard))

Different techniques for solving equation system (manual, ..)

Application to Thalys example

$\underline{N}^T \, \underline{c}_R$ **= 0,** with $\underline{N}^T$=

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}
t_1	1	-1							-1				1
t_2		1	-1										
t_3			1	-1									
t_4				1	-1						1		
t_5					1	-1	-1					1	
t_6	-1					1							
t_7							1	-1					
t_8								1				-1	
t_9									1	-1			
t_{10}										1	-1		-1

Solutions? Educated guessing:

$\sum_{\text{rows}}=0 \rightarrow$ 1 linear dependency among rows $\rightarrow$ rank = 10-1 = 9

Dimension of solution space = 13 - rank = 4

4 components of (6, 11, 12, 13) of $\underline{c}_R$ are independent

$\rightarrow$ set one of these to 1

and others to 0 to obtain a basis for the solution space

1^{st} basis

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}
t_1	1	-1							-1				1
t_2		1	-1										
t_3			1	-1									
t_4				1	-1						1		
t_5					1	-1	-1					1	
t_6	-1					1							
t_7							1	-1					
t_8								1				-1	
t_9									1	-1			
t_{10}										1	-1		-1

Set one of components (6, 11, 12, 13) to 1, others to 0.
$\rightarrow$ **1st basis** b_1:
$b_1(s_6)=1$, $b_1(s_{11})=0$, $b_1(s_{12})=0$, $b_1(s_{13})=0$

- $t_{10}(s_{10})\, b_1(s_{10}) + t_{10}(s_{11})\, b_1(s_{11}) + t_{10}(s_{13})\, b_1(s_{13}) = 0$
  $\rightarrow b_1(s_{10}) = 0$
- $t_9(s_9)\, b_1(s_9) + t_9(s_{10})\, b_1(s_{10}) = 0$
  $\rightarrow b_1(s_9)=0$
- $b_1 \doteq (1,1,1,1,1,1,0,0,0,0,0,0,0)$

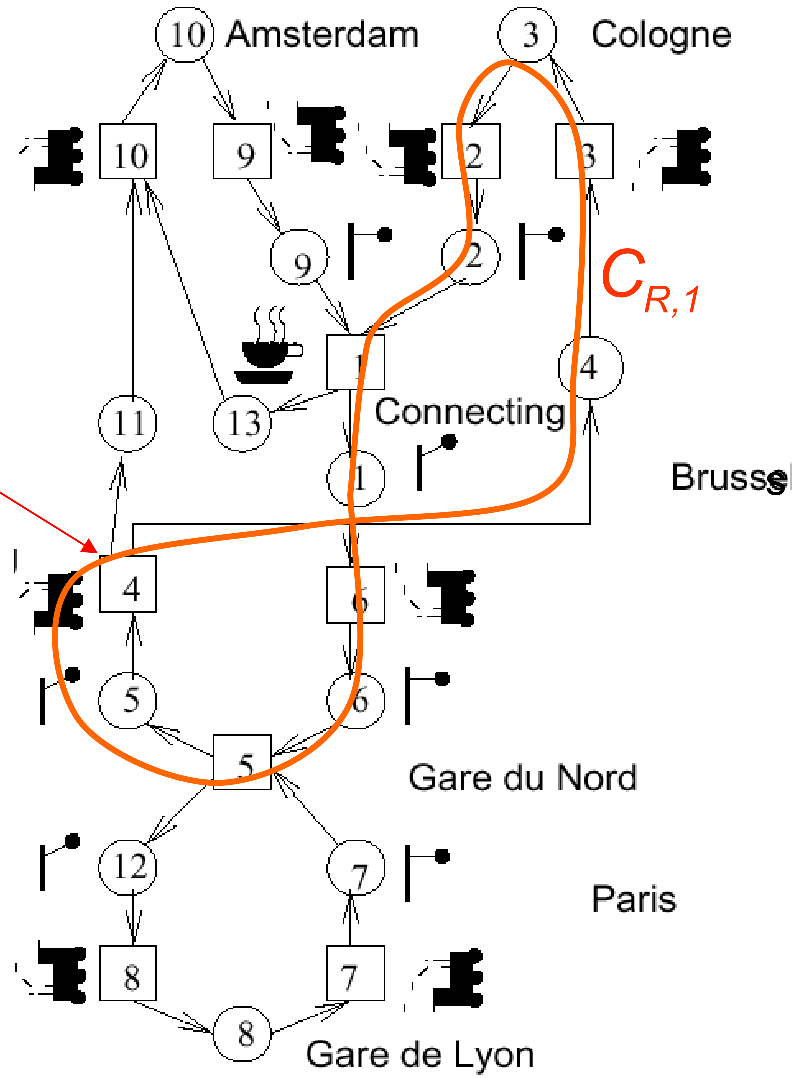
All components $\in \{0, 1\}$
$\rightarrow \boldsymbol{c_{R1}} = b_1$

Interpretation of the 1st invariant

$$c_{R,1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Characteristic vector describes places for Cologne train.

We proved that: the number of trains along the path remains constant.



2^{nd} basis

	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}
t_1	1	-1							-1				1
t_2		1	-1										
t_3			1	-1									
t_4				1	-1						1		
t_5					1	-1	-1					1	
t_6	-1					1							
t_7							1	-1					
t_8								1				-1	
t_9									1	-1			
t_{10}										1	-1		-1

Set one of components (6, 11, 12, 13)
to 1, others to 0.

$\rightarrow$ **2nd basis** b_2:

$b_2(s_6)=0$, $b_2(s_{11})=1$,

$b_2(s_{12})=0$, $b_2(s_{13})=0$

- $t_{10}(s_{10})\, b_2(s_{10}) + t_{10}(s_{11})\, b_2(s_{11}) + t_{10}(s_{13})\, b_2(s_{13}) = 0$

  $\rightarrow b_2(s_{10}) = 1$
- $t_9(s_9)\, b_2(s_9) + t_9(s_{10})\, b_2(s_{10}) = 0$

  $\rightarrow b_2(s_9)=1$
- ...

$b_2 = (0,-1,-1,-1,0,0,0,0,1,1,1,0,0)$

$b_1 = (1, 1, 1, 1,1,1,0,0,0,0,0,0,0)$

b_2 not a characteristic vector, but $\boldsymbol{c}_{R,2}=b_1+b_2$ is

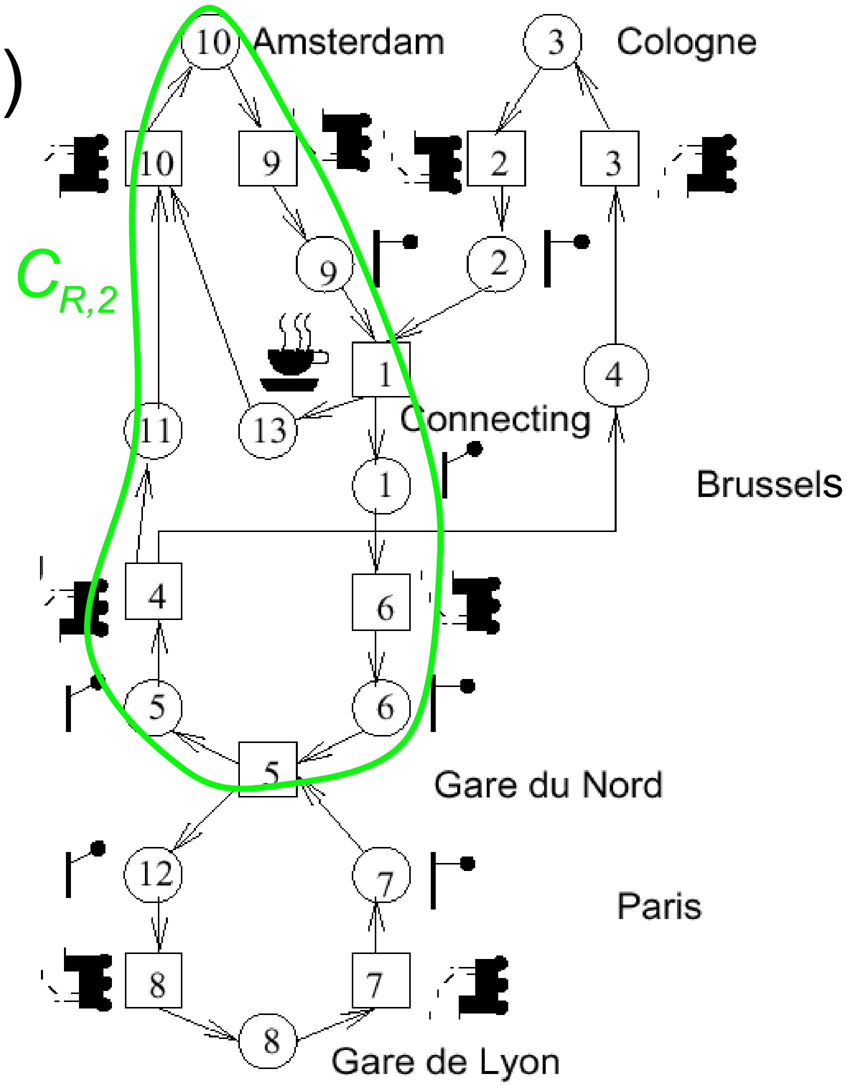
$\rightarrow \boldsymbol{c}_{R,2}$ =
(1,0,0,0,1,1,0,0,1,1,1,0,0
)

Interpretation of the 2nd invariant

$c_{R,2} = (1,0,0,0,1,1,0,0,1,1,1,0,0)$

We proved that:

None of the Amsterdam trains gets lost (nice to know ☺).



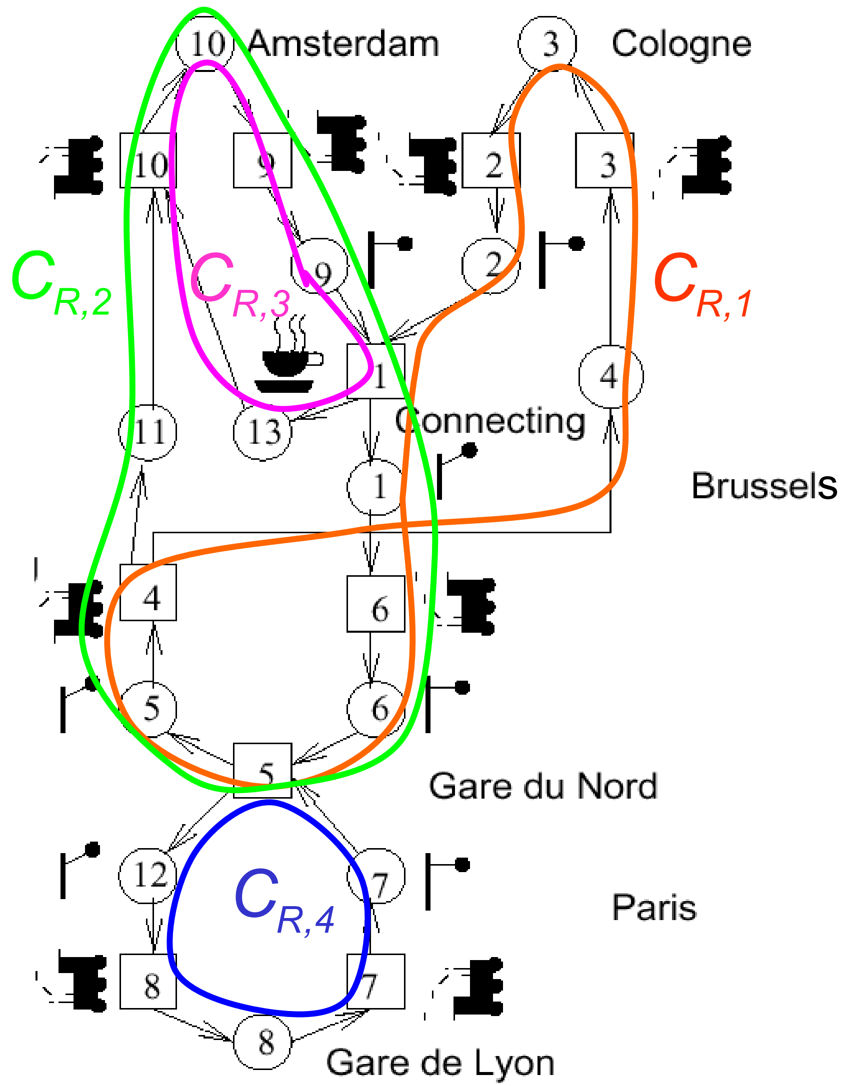
Setting $b_3(s_{12})$ to 1 and $b_4(s_{13})$ to 1 leads to an additional 2 invariants

$$c_{R,1} = \begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$c_{R,2} = \begin{pmatrix}1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0\end{pmatrix}$$

$$c_{R,3} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1\end{pmatrix}$$

$$c_{R,4} = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0\end{pmatrix}$$



We proved that:

- the number of trains serving Amsterdam, Cologne and Paris remains constant.
- the number of train drivers remains constant.

Applications

- Modeling of resources;
- modeling of mutual exclusion;
- modeling of synchronization.

Predicate/transition nets

Goal: compact representation of complex systems.

Key changes:

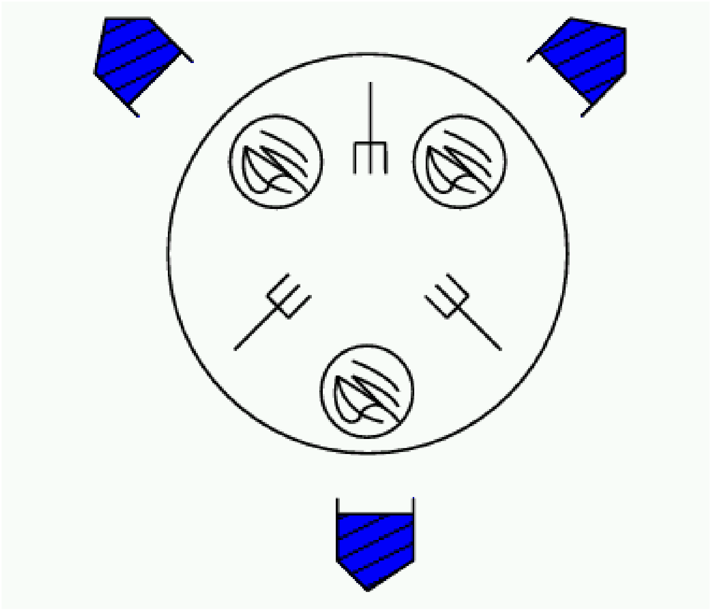
- Tokens are becoming individuals;
- Transitions enabled if functions at incoming edges true;
- Individuals generated by firing transitions defined through functions

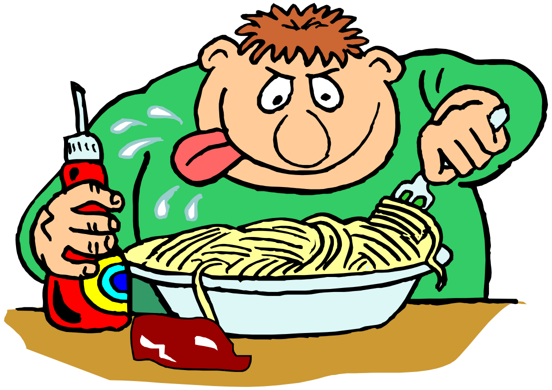
Changes can be explained by folding and unfolding C/E nets,

☞ semantics can be defined by C/E nets.

Example: Dining philosophers problem

n>1 philosophers sitting at a round table;
n forks,
n plates with spaghetti;
philosophers either thinking or eating spaghetti
(using left and right fork).





2 forks needed!

How to model conflict for forks?

How to guarantee avoiding starvation?

Condition/event net model of the dining philosophers problem

Let $x \in \{1..3\}$

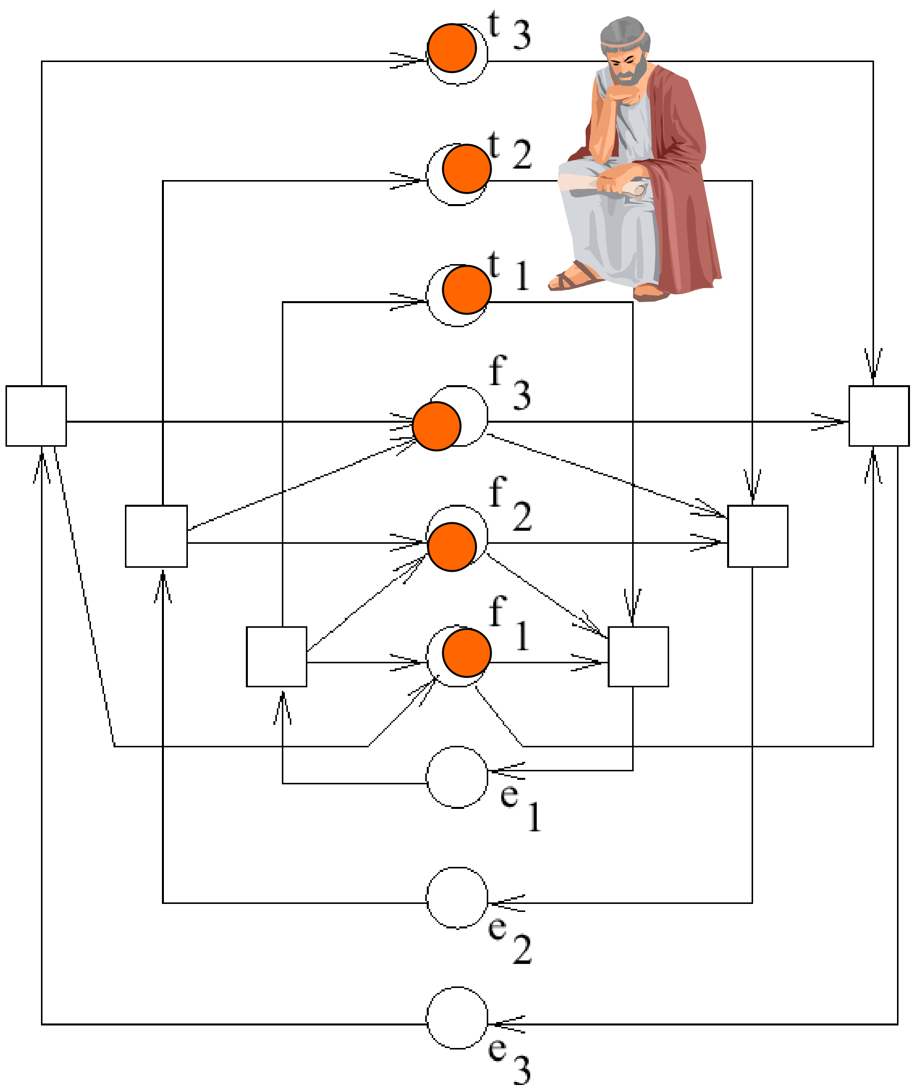
t_x: x is thinking

e_x: x is eating

f_x: fork x is available

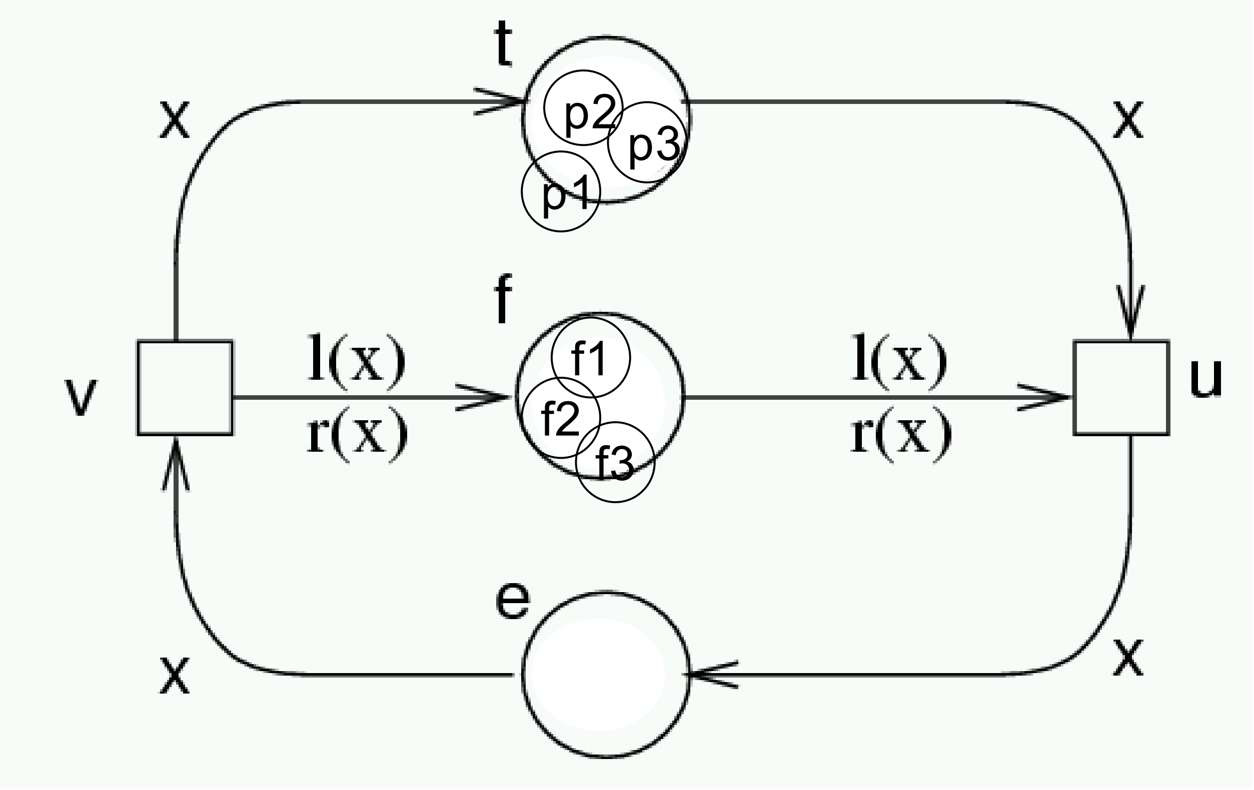
Model quite clumsy.

Difficult to extend to more philosophers.



Predicate/transition model of the dining philosophers problem (1)

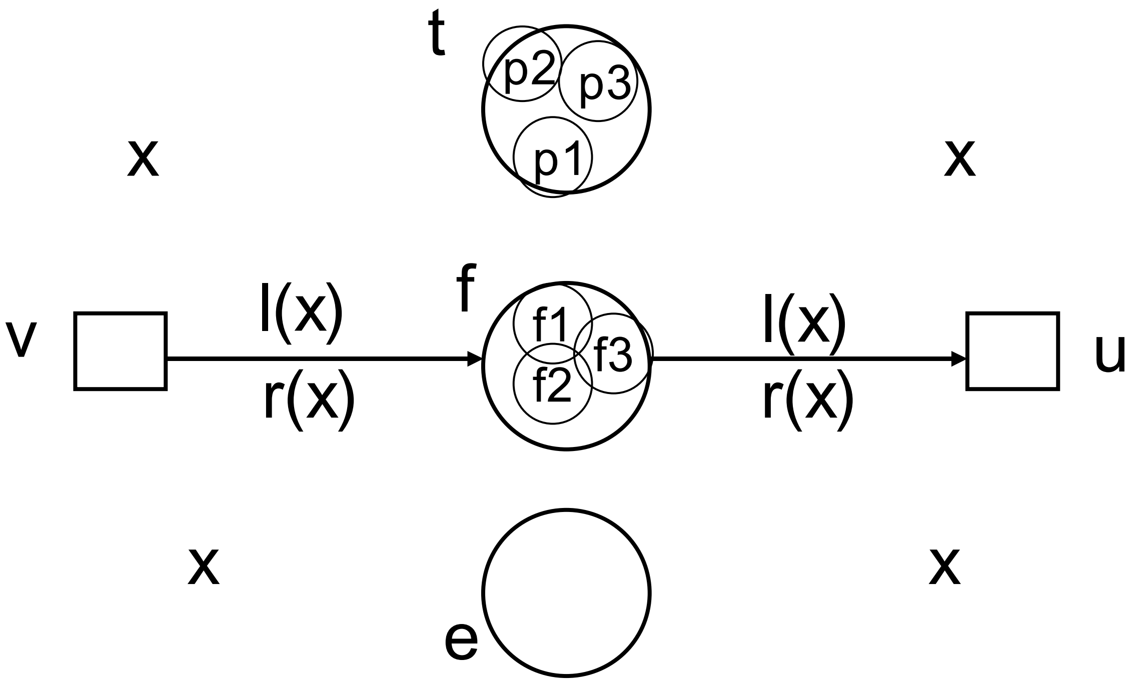
Let x be one of the philosophers,
let l(x) be the left spoon of x,
let r(x) be the right spoon of x.



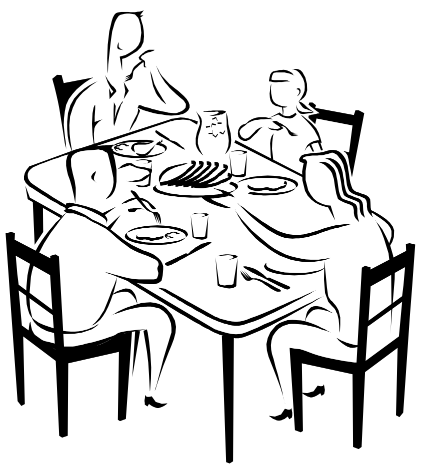
Tokens: individuals.

Semantics can be defined by replacing net by equivalent condition/event net.

Predicate/transition model of the dining philosophers problem (2)



Model can be extended to arbitrary numbers of people.



Evaluation

Pros:

- Appropriate for distributed applications,
- Well-known theory for formally proving properties,
- Initially a quite bizarre topic, but now accepted due to increasing number of distributed applications.

Cons (for the nets presented) :

- problems with modeling timing,
- no programming elements,
- no hierarchy.

Extensions:

- Enormous amounts of efforts on removing limitations.

Summary

Petri nets: focus on causal relationships

Condition/event nets

- Single token per place

Place/transition nets

- Multiple tokens per place

Predicate/transition nets

- Tokens become individuals
- Dining philosophers used as an example

Extensions required to get around limitations