Appendix on Mathematics

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Utilization Bounds: (slides set 2 and many others)

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then their arithmetic mean is larger than or equal to the geometric mean. The tightness happens when $a_1 = a_2 = \cdots = a_n$. Therefore,

$$\frac{\sum_{i=1}^{n} a_i}{n} \geq \sqrt[n]{\prod_{i=1}^{n} a_i}.$$

Let $a_i = U_i + 1$. Therefore, we know that

$$\left(\frac{\sum_{i=1}^{n} U_i}{n} + 1\right)^n \geq \prod_{i=1}^{n} (1 + U_i).$$

The definition of $(1 + \frac{x}{n})^n$ is e^x when $n \to \infty$, whereas $(1 + \frac{x}{n})^n \leq e^x$ for $n \geq 0$.

Utilization Bounds: (slides set 2 and many others) cont.

Therefore, if $\prod_{i=1}^{n}(1+U_i) > y$, this means that

$$e^{\sum_{i=1}^{n} U_i} \geq \prod_{i=1}^{n}(1+U_i) > y.$$

Then $\sum_{i=1}^{n} U_i > \ln y$.

Integer linear programming models

Ingredients: (involving linear expressions of integer variables from a set $\mathbf{X}$)

- Cost function: $C = \sum_{x_i \in \mathbf{X}} a_i x_i$ with $a_i \in \mathbb{R}, x_i \in \mathbb{N}$
- Constraints: $\forall j \in \mathbf{J} : \sum_{x_i \in \mathbf{X}} b_{i,j} x_i \geq c_j$ with $b_{i,j}, c_j \in \mathbb{R}$

Definition: The problem of minimizing the cost function subject to the constraints is called an integer linear programming (ILP) problem.

- If all x_i are constrained to be either 0 or 1, the ILP problem is said to be a 0/1 integer linear programming problem.
- The case of $x_i \in \mathbb{R}$ is called linear programming (LP).

Example

$$C = 5x_1 + 6x_2 + 4x_3$$

$$x_1 + x_2 + x_3 \geq 2$$

$$x_1, x_2, x_3 \in \{0,1\}$$

x_1	x_2	x_3	C
0	1	1	10
1	0	1	9 ← Optimal
1	1	0	11
1	1	1	15

Remarks on ILP and LP

- Maximizing the cost function: just set $C = -C$
- Integer programming is NP-complete.
- Running times depend exponentially on problem size, but problems of ¿1000 vars solvable with good solver (depending on the size and structure of the problem)
- The case of some $x_i \in \mathbb{R}$ and some $x_i \in \mathbb{N}$ is called mixed integer-linear programming.
- ILP/LP models good starting point for modeling, even if heuristics are used in the end.
- Solvers: lp_solve (public), CPLEX (commercial)

Knapsack Problem

The Knapsack problem:

- You can select items to put in your backpack with the maximum total value.
  - There are m items available.
  - Item i weights w_i kg,
  - Item i has value v_i.
  - You can carry Q kg.

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$$
\begin{aligned}
\text{maximize} \sum_i x_i v_i & \\
\text{s.t.} \sum_i x_i w_i \leq Q & \\
x_i \in \{0, 1\}, & \qquad \forall i
\end{aligned}
$$

Fractional Knapsack Problem

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$$\text{maximize} \sum_i x_i v_i$$
$$\text{s.t.} \sum_i x_i w_i \leq Q$$
$$0 \leq x_i \leq 1, \qquad \forall i$$

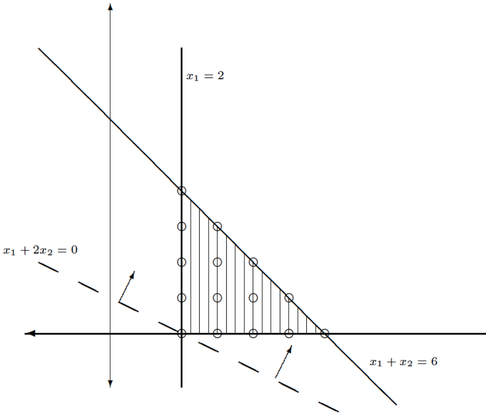
Extreme Point Theory of LP

The solution of an LP forms a polyhedron. An extreme point is an intersection of the linear constraints.

- constraints: $x_1 \geq 2$,
  $x_1 + x_2 \leq 6$,
  $x_1 + 2x_2 \geq 0$.
- Intersections: 3 points

The extreme point theory:

- either the polyhedron is unbounded, or
- the polyhedron is bounded and the optimal solution lies in one of the extreme points.



Simplex Algorithm

- Developed by Prof. Dantzig.
- The basic idea is to search among the extreme points.
- It starts from the so-called basic feasible solution (as Phase I)
- It then iterates to visit the extreme points by going through the edges. If an infinite edge is found, it implies that the polyhedron is unbounded and the optimal solution is unbounded.